$\frac{59}{60}$ 93

RS

$\mathrm{I} = \frac{12}{12}$

$\mathrm{II} = \frac{47}{48}$

Dan Ports

18.03 PS5

I. 4.6.1. $\ddot{x} + 4x = 0,\ \dot{x}(0) = 1$

$(s+2i)(s-2i) = 0 \Rightarrow s = \pm 2i$

$x = a\cos 2t + b\sin 2t, \quad x'(0) = 2a \Rightarrow a = \frac{1}{2}$

$x = \frac{1}{2}\cos 2t$

1.7.2. $\dot{x} + x = \sin(t) \Rightarrow x_p = \mathrm{Im}\left(\frac{e^{it}}{1+i}\right) = \mathrm{Im}\left(\frac{e^{it}}{2}(1-i)\right)$

$= \frac{1}{2}(\sin t - \cos t), \quad x_h = ce^{-t}$

$x(0) = 0 \Rightarrow c = \frac{1}{2}$

$x = \frac{1}{2}\sin t - \frac{1}{2}\cos t + \frac{1}{2}e^{-t}, \quad t < \pi$

$x(\pi) = \frac{1}{2} + \frac{1}{2}e^{-\pi} \quad x(\pi) = \frac{3}{2} + \frac{1}{2}e^{-\pi}$

$x = \frac{1}{2}\sin t - \frac{1}{2}\cos t + (\frac{1}{2} + e^{\pi})e^{-t}, \quad t > \pi$

6.1.5 $y' - y = -x - 1 \Rightarrow y = ax + b$

$a - ax + b = -x - 1 \Rightarrow a = 1, \quad a = 2 \Rightarrow y = x + 2 - e^{x}$

x	h = 0.1	h = 0.05	actual
.1	1	.998	.994
.2	.99	.981	.979
.3	.968	.960	.950
.4	.936	.922	.909
.5	.890	.871	.851

6.1.9. $y = \tan(\frac{1}{4}t + \frac{1}{4}\pi)$

t	h = .1	h = 0.05	act
.1	1.05	1.05	1.05
.2	1.10	1.10	1.11
.3	1.16	1.16	1.16
.4	1.21	1.22	1.22
.5	1.27	1.28	1.29

4.4.1 $\int_0^t \tau \, d\tau = t^2/2$

4.4.3 $\int_0^t (\sin\tau)\cdot\sin(t-\tau)\, d\tau = 1/4(\sin 2\tau - t) - 1/2\, 2t \cos t \big|_0^t$

$= \dfrac{-t\cos t}{2} - \dfrac{\sin t}{2}$

4.4.5 $\int_0^t e^{a\tau}\cdot e^{a(t-\tau)}\, d\tau = \int_0^t e^{at}\, d\tau =$

$= \tau e^{at}\big|_0^t = te^{at}$

4.6.9, $x'' + 4x \Rightarrow s^2 + 4 = 0 \Rightarrow (s+2i)(s-2i)$

$\Rightarrow x_h = a\cos 2t + b\sin 2t$, $a=0$, $b = 1/2$

$x = \int_0^t 1/2 \sin\tau)f(t-\tau)\, d\tau$

4.6.11: $x'' + 6x' + 8x \Rightarrow s^2 + 6s + 8 = 0 \Rightarrow s = -4,\ s = -2$

$\Rightarrow x_h = ae^{-4t} + be^{-2t}$, $a + b = 0$

$-4a - 2b = 1 \Rightarrow -4a + 2a = -2a = 1$

$x_h = -1/2 e^{-4t} + 1/2 e^{-2t}$

$x = \int_0^t (-1/2 e^{-4\tau} + 1/2 e^{-2\tau}) f(t-\tau)\, d\tau$

3A/1. $\mathcal{L}(t) = \int_0^\infty te^{st}\, dt$, $u = t$, $du = dt$

$dv = e^{st}$, $V = 1/s\, e^{st}$

$= t/s\, e^{st}\big|_0^\infty - \int_0^\infty 1/s\, e^{st}\, dt$

$= 0 - 0 - (1/s^2\, e^{st})\big|_0^\infty$

$= 0 - (0 - 1/s^2) = 1/s^2$

3A/3. a. $\mathcal{L}^{-1}(1/(s/2+3)) = \mathcal{L}^{-1}\left(2\cdot\left(\frac{1}{s+6}\right)\right) = 2\cdot e^{-6t}$

b. $\mathcal{L}^{-1}\left(\dfrac{3}{s^2+4}\right) = \dfrac{3}{2}\mathcal{L}^{-1}\left(\dfrac{2}{s^2+2^2}\right) = \dfrac{3}{2}\sin(2t)$

3A / 3C $\frac{1}{s^2-4} = \frac{a}{s-2} + \frac{b}{s+2} = \frac{1/4}{s-2} - \frac{1/4}{s+2}$

$\mathcal{L}^{-1}\left(\frac{1}{s^2-4}\right) = \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) - \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right)$

$= \frac{1}{4}e^{2t} - \frac{1}{4}e^{-2t}$

$\mathcal{L}^{-1}\left(\frac{1+2s}{s^3}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)$

$= \frac{1}{2}t^2 + 2t$

e $\frac{1}{s^4-9s^2} = \frac{a}{s^2} + \frac{b}{s} + \frac{c}{s-3} + \frac{d}{s+3}$

$= \frac{-1}{9}\frac{1}{s^2} + \frac{0}{s} + \frac{1}{54}\frac{1}{(s-3)} - \frac{1}{54(s+3)}$

$\mathcal{L}^{-1} = -\frac{1}{9}\mathcal{L}^{-1}(1/s^2) + \frac{1}{54}\left(\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+3}\right)\right)$

$= \frac{-1}{9}t + \frac{1}{54}\left(e^{3t} - e^{3t}\right)$

II.

1a. $Q(t) = 100 + \sum_{n=0}^{\infty} 0.01 \cdot u(t-n)$

$q(t) = \sum_{n=0}^{\infty} 0.01 \, \delta(t-n)$

The closest continuous approximation gives $Q(t) = 100.01 + 0.01t$,
$q(t) = 0.01$

b. for $t < \pi$, $\dot{x} + x = \sin t$

$\Rightarrow \dot{z} + z = e^{it} \Rightarrow z_p = \frac{e^{it}}{(1+i)} \quad e^{it}(1-i)/2$

$\Rightarrow x_p = (\sin t - \cos t)/2$

$x_h = ce^{-t}$; $x = (\sin t - \cos t)/2 + ce^{-t}$

$x(0) = 1 \Rightarrow -1/2 + c = 1 \Rightarrow c = 3/2$

$x = \frac{1}{2}(\sin t - \cos t) + 3/2\, e^{-t}$, $t < \pi$

At $t = \pi$, $x(\pi^-) = 1/2 + 3/2\, e^{-\pi}$

$x(\pi^+) = x(\pi^-) + 1 = 3/2 + 3/2\, e^{-\pi}$

For $t > \pi$, $x = 1/2(\sin t - \cos t) + ce^{-t}$

$x(\pi^+) = 3/2 + 3/2\, e^{-\pi} = 1/2 + ce^{-\pi}$

$\Rightarrow 1 + 3/2\, e^{-\pi} = ce^{-\pi} \Rightarrow e^{\pi} + 3/2 = c$

$x = 1/2(\sin t - \cos t) + (e^{\pi} + 3/2)e^{-t}$

c. $w(t) = e^{-at} \Rightarrow \boxed{\dot{x} + ax = \delta(t)}$

$w(t) = e^{-t/2} \sin t \Rightarrow w(t) = \mathrm{Im}(e^{-t/2 + it})$

The eqn is a 2nd-order ODE whose characteristic polynomial has roots $-1/2 \pm i$

$p(s) = (s + 1/2 + i)(s + 1/2 - i) = s^2 + s + 5/4$

$\Rightarrow \boxed{\ddot{x} + \dot{x} + 5/4\, x = \delta(t)}$

$w(t) = u(t) \Rightarrow \boxed{\dot{x} = \delta(t)}$

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20. $n=1 \Rightarrow x(1) = 1+1$

$n=2 \Rightarrow x(1) = 1 + 1/2 + 1/2(1+1/2) = 1 + 2/2 + 1/4 = 9/4$

$n=3 \Rightarrow x(1) = 1 + 1/3 + 1/3(1+1/3) + 1/3(1+1/3+1/3+1/9)$

$= 1 + 3/3 + 3/9 + 1/27 = 2 + 1/3 + 1/27$

$= (1+1/3)^3 = 64/27$

For arbitrary n, $x(1)$ is approximated as $\boxed{(1+1/n)^n}$ applying the binomial expansion above.

ok

$\lim_{n\to\infty} x(1) = \lim_{n\to\infty} (1+1/n)^n = (\frac{n+1}{n})^n = L$

$\Rightarrow \lim_{n\to\infty} n \ln(\frac{n+1}{n}) = \ln L = \lim_{n\to\infty} \ln(\frac{n+1}{n})/1/n$

$= \lim_{n\to\infty} \frac{\frac{n}{n+1}(-1/n^2)}{-1/n^2} = \lim_{n\to\infty} \frac{n}{n+1} = 1$

$\Rightarrow \ln L = 1 \Rightarrow \boxed{L = e}$

The estimate at $n=1000$ differs from e by $\boxed{0.001\%}$ approximately

21a. $f(t) * w(t) = \int_0^t (1+\cos(b\tau))e^{-a(t-\tau)}d\tau$

$= \int_0^t e^{-a(t-\tau)}d\tau - e^{-at}\int_0^t e^{a\tau}\cos(b\tau)d\tau$

$= Re[e^{-at}\int_0^t e^{a\tau}d\tau - e^{-at}\int_0^t e^{(a+bi)\tau}d\tau]$

$= Re[e^{-at}(\frac{1}{a}e^{a\tau}|_0^t - \frac{1}{a+bi}e^{(a+bi)\tau}|_0^t)]$

$= Re[e^{-at}(\frac{1}{a}e^{at} - 1/a - (\frac{1}{a+bi}e^{(a+bi)t} - \frac{1}{a+bi})]$

$= Re[\frac{1}{a} - \frac{1}{a}e^{-at} + \frac{(a-bi)}{a^2+b^2}(e^{(a+bi)t} - 1)]$

$\boxed{= \frac{1}{a} - \frac{1}{a}e^{-at} - \frac{a}{a^2+b^2}e^{-at} + \frac{1}{a^2+b^2}(a\cos(bt) + b\sin(bt))}$

b. See attached.

21.c. $\dot{x} + ax = 1 + \cos(bt)$, $x(0) = 0$

$x_h = Ce^{-at}$, $x_{p_1} = 1/a$

$\dot{z} + az = e^{bit} \Rightarrow z_p = \frac{e^{bit}}{(a+bi)} = \frac{e^{bit}(a-bi)}{a^2+b^2}$

$\Rightarrow x_{p_2} = \frac{a\cos(bt) + b\sin bt}{a^2+b^2}$

$x = x_{p_1} + x_{p_2} + x_h = 1/a + \frac{a\cos(bt) + b\sin(bt)}{a^2+b^2} + Ce^{-at}$

$x(0) = 0 \Rightarrow \frac{1}{a} + \frac{a}{a^2+b^2} + C = 0 \Rightarrow C = -1/a - \frac{a}{a^2+b^2}$

$\boxed{x = 1/a + \frac{a\cos(bt) + b\sin(bt)}{a^2+b^2} + \left(-1/a - \frac{a}{a^2+b^2}\right)e^{-at}}$

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The periodic component can be written

$\frac{1}{\sqrt{a^2+b^2}} \cos(bt - \arctan(b/a))$

$A \neq 1$

21d. $f(t) * 1 = \int_0^t (1 + \cos b\tau) \cdot 1 \, d\tau$

$= [\tau + 1/b \sin \tau]_0^t = \boxed{t + 1/b \sin bt}$

$\dot{x} + 0x = 1 + \cos bt \Rightarrow x = \int 1 + \cos bt \, dt$

$= t + 1/b \sin bt$

22.a $\mathcal{L}[f(at); s] = \int_0^\infty f(at) e^{-st} dt$

Designate $u = at \Rightarrow du = a\,dt$

$\int_0^\infty f(at) e^{-st} dt = 1/a \int_0^\infty f(u) e^{-s/a\, u} du.$

With a change of variables, this is like $F(s)$ except with a factor $1/a$ and the $-s/a$ instead of $-s$

Therefore $\mathcal{L}[f(at); s] = \boxed{1/a \, F(s/a)}$

Check: $\mathcal{L}(e^t) = 1/(s-1)$

$\mathcal{L}(e^{at}) = 1/a \left[\frac{1}{s/a - 1}\right] = \frac{1}{s-a}$, as expected.

22b. $\mathcal{L}(\cos(wt); s) = \frac{s}{s^2+w^2}$

$\mathcal{L}(\sin(wt); s) = w/(s^2+w^2)$

Using s-shifting $\boxed{\mathcal{L}(e^{at}\cos wt; s) = \frac{(s-a)}{(s-a)^2+w^2}}$

and $\boxed{\mathcal{L}(e^{at}\sin wt; s) = \frac{w}{(s-a)^2+w^2}}$

c. $\mathcal{L}(t, s) = \int_0^\infty t e^{-st} = \left[-t/s\, e^{-st} + \int -1/s\, e^{-st}\,dt\right]_0^\infty$

$= \left[-t/s\, e^{-st} + 1/s^2\, e^{-st}\right]_0^\infty = (0+0)-(0-1/s^2)$

$= 1/s^2$

$\cos wt = \frac{e^{iwt}+e^{-iwt}}{2}$

$t\cos wt = t/2\, e^{iwt} + t/2\, e^{-iwt}$

$\Rightarrow \mathcal{L}(t\cos wt; s) = \frac{1}{2}(1/(s-iw)^2) + 1/2\,(1/(s+iw)^2)$

$= \frac{s^2-w^2}{(s^2+w^2)^2}$

$t\sin(wt) = t/2i\, e^{iwt} - t/2i\, e^{-iwt}$

$\Rightarrow \mathcal{L}(t\sin(wt); s) = \frac{1}{2i}\left(\frac{1}{(s-iw)^2} - \frac{1}{(s+iw)^2}\right)$

$= \frac{2sw}{(s^2+w^2)^2}$

$\mathcal{L}(\cos(wt); s) = \frac{s}{s^2+w^2} = \frac{s(s^2+w^2)}{(s^2+w^2)^2}$

$\mathcal{L}(\sin(wt); s) = \frac{w}{s^2+w^2} = \frac{w(s^2+w^2)}{(s^2+w^2)^2} = \frac{ws^2+w^3}{(s^2+w^2)^2}$

$\mathcal{L}(t\sin(wt); s) = \frac{s^2-w^2}{(s^2+w^2)^2}$

$\mathcal{L}(t\cos(wt); s) = \frac{2sw}{(s^2+w^2)^2}$

To obtain $\frac{s}{(s^2+w^2)^2}$, divide $\frac{2sw}{(s^2+w^2)^2}$ by $2w$.

Therefore, $\boxed{\frac{s}{(s^2+w^2)^2} = \mathcal{L}\left(\frac{1}{2w} t\sin(wt)\right)}$

To obtain $\frac{1}{(s^2+w^2)^2}$, use $\frac{ws^2+w^3 - w(s^2-w^2)}{(2w^2)\cdot(s^2+w^2)^2}$

Therefore,

$$\frac{1}{(s^2+w^2)^2} = \mathcal{L}\left[\frac{1}{2w^2}\left(\sin(wt) - wt\cos(wt)\right)\right]$$

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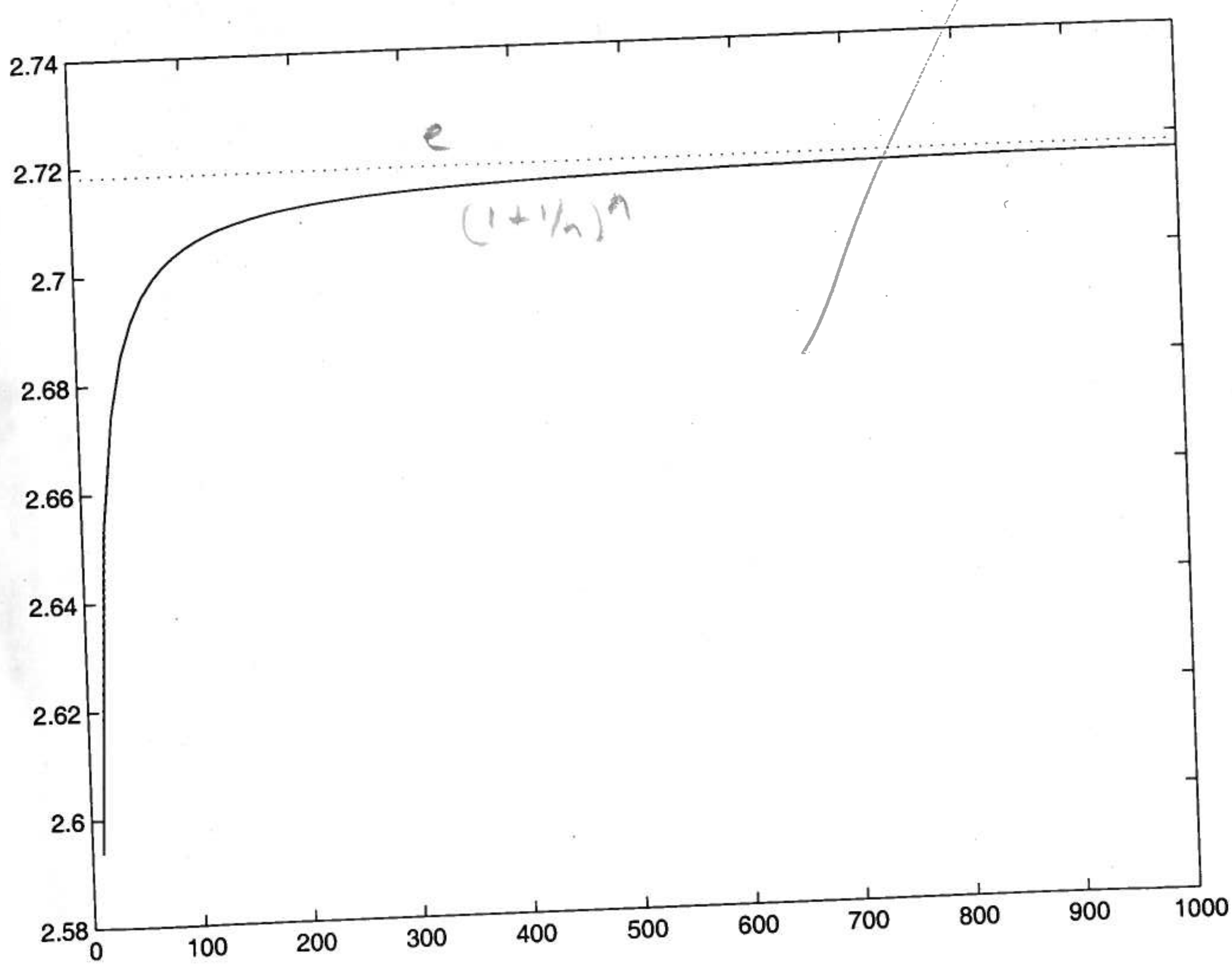
II. 20



2.74
2.72
2.7
2.68
2.66
2.64
2.62
2.6
2.58
0
100
200
300
400
500
600
700
800
900
1000
e
(1+1/n)^n

II. 21