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18.06 PS10

Part 1: 15/15
Part 2: 16/19
Grade: 9

I 10A.

5

$$\det(P_L - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 1 & 1-\lambda & 0 & 0 \\ 1 & 2 & 1-\lambda & 0 \\ 1 & 3 & 3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$$

$$\det(P_L^{-1} - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ -1 & 1-\lambda & 0 & 0 \\ 1 & -2 & 1-\lambda & 0 \\ -1 & 3 & -3 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)(1-\lambda)(1-\lambda)(1-\lambda) = 0 \Rightarrow \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$$

P_L and P_L^{-1} have the same eigenvalues.

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad D^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} = D$$

$\therefore P_L^{-1} = D^{-1} P_L D \Rightarrow P_L$ similar to P_L^{-1}

$$P_L D = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ 1 & 2 & 1 & \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & -1 & & \\ 1 & -2 & 1 & \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

$$(P_L D)^{-1} = D^{-1} P_L^{-1} = D P_L^{-1}$$

$$= \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & 1 & \\ & & & -1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ 1 & -2 & 1 & \\ -1 & 3 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ 1 & -1 & & \\ 1 & -2 & 1 & \\ 1 & -3 & 3 & -1 \end{bmatrix}$$

$$= P_L D.$$

So $P_L D$ is its own inverse.

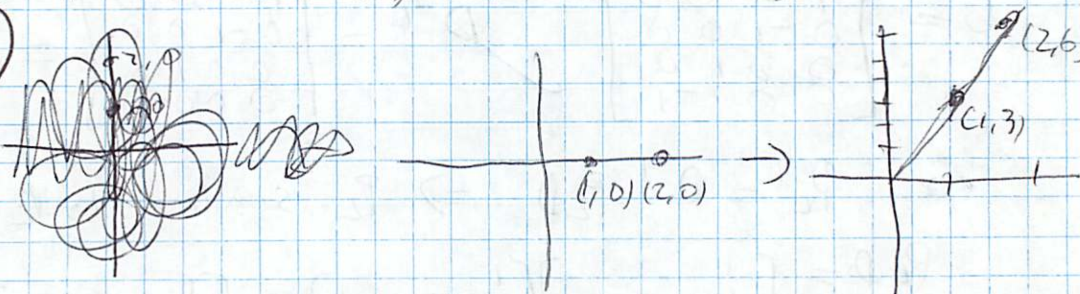
$$P_L - 1I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 3 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_n \end{bmatrix} = \vec{0}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 3 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}.$$

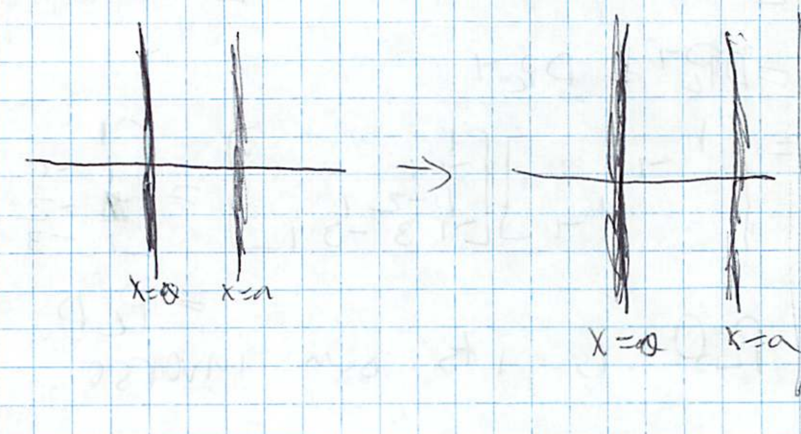
This matrix has 3 pivots, so its nullspace is 1-dimensional. Since the $\lambda = 1$ eigenspace of P_L is 1-dimensional and $\lambda = 1$ is the only eigenvalue, there is only one independent eigenvector of P_L. So the Jordan form

$$J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

10B. $T(1,0) = (1,3)$; $T(2,0) = (2,6)$



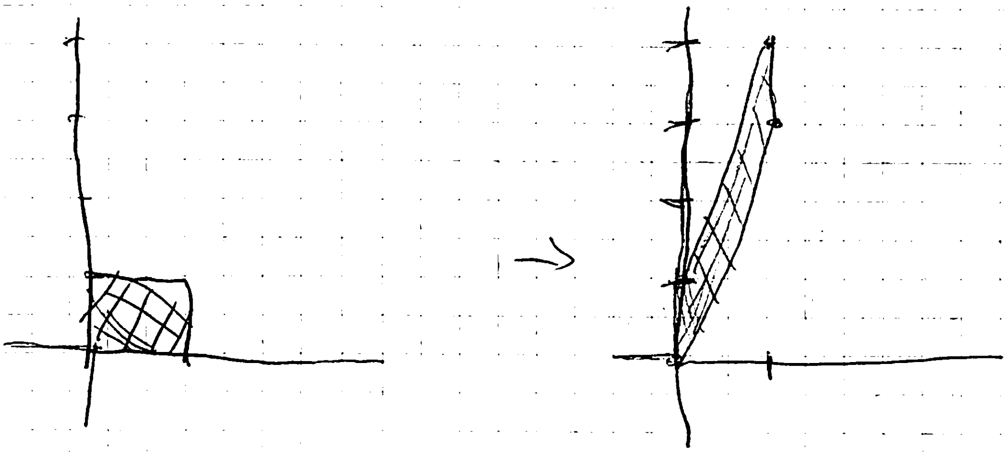
$T(0,y) = (0,y)$; $T(a,y) = (a, 3a+y)$



Each point on the $x=0$ line is unchanged. Each point on the $x=a$ line is shifted upward by $3a$ units

$T(0,0)=(0,0)$ $\quad$ $T(0,1)=(0,1)$

$T(1,0)=(1,3)$ $\quad$ $T(1,1)=(1,4)$



c. $T(v_1)=\begin{bmatrix}0&0\\0&1\end{bmatrix}=u_4$ $\quad$ $T(v_2)=\begin{bmatrix}0&0\\1&0\end{bmatrix}=u_3$

(3) $T(u_3)=\begin{bmatrix}0&1\\0&0\end{bmatrix}=u_2$ $\quad$ $T(v_4)=\begin{bmatrix}1&0\\0&0\end{bmatrix}=u_1$

$$A=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix} \qquad A^{-1}=\begin{bmatrix}0&0&0&1\\0&0&1&0\\0&1&0&0\\1&0&0&0\end{bmatrix}=A$$

$\Rightarrow T^{-1}=T$ $\quad$ $T=?$?

Assuming that this problem means that T_2 multiplies its input by $M=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ on the left.

$$T_2(u_1)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}a&0\\c&0\end{bmatrix}=au_1+cu_3$$

$$T_2(u_2)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0&1\\0&0\end{bmatrix}=\begin{bmatrix}0&a\\0&c\end{bmatrix}=au_2+cu_4$$

$$T_2(u_3)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix}=\begin{bmatrix}b&0\\d&0\end{bmatrix}=bu_1+du_3$$

$$T_2(u_4)=\begin{bmatrix}a&b\\c&d\end{bmatrix}\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&b\\0&d\end{bmatrix}=bu_2+du_4$$

$$\Rightarrow A_2=\begin{bmatrix}a&0&b&0\\0&a&0&b\\c&0&d&0\\0&c&0&d\end{bmatrix}$$

10D 2 Let R be the matrix whose rows are a basis for the row space. R is $r \times n$:

$$R = \begin{bmatrix} - v_1 - \\ - v_2 - \\ - v_r - \end{bmatrix}$$

Let C be the matrix whose columns are a ~~matrix~~ basis for the column space. C is $m \times r$, $C = \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \dots & u_r \\ | & | & & | \end{bmatrix}$

Multiplying CR gives a matrix $m \times n$ that has the correct row and column space bases.

~~To find all such matrices, multiply CRB, where B is a rank r mxn matrix. This gives~~

To find all such matrices A, multiply $A = CBR$, where B is any $r \times r$ matrix. The resulting $m \times n$ matrix has the correct bases for its row and column space. Since we want A to have rank r, we must also require B to have rank r.

II. 6.6.3 (+)

a. $\underset{M^{-1}}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\underset{A}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}\boxed{\underset{M}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}} = \underset{B}{\begin{bmatrix}0&0\\0&1\end{bmatrix}}$

b. $\underset{M^{-1}}{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}\underset{A}{\begin{bmatrix}1&1\\1&1\end{bmatrix}}\boxed{\underset{M}{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}} = \underset{B}{\begin{bmatrix}1&-1\\-1&1\end{bmatrix}}$ ✓

c. $\underset{M^{-1}}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}\underset{A}{\begin{bmatrix}1&2\\3&4\end{bmatrix}}\boxed{\underset{M}{\begin{bmatrix}0&1\\1&0\end{bmatrix}}} = \underset{B}{\begin{bmatrix}4&3\\2&1\end{bmatrix}}$

666 (+)

$J = \begin{bmatrix}1&0\\0&0\end{bmatrix} \Rightarrow tr=1,\ det=0 \Rightarrow \begin{bmatrix}1&\\&0\end{bmatrix}$ or $\begin{bmatrix}0&\\&1\end{bmatrix}$ on diagonal and $\begin{bmatrix}&1\\0&\end{bmatrix}, \begin{bmatrix}&1\\0&\end{bmatrix}$ or $\begin{bmatrix}&0\\0&\end{bmatrix}$ on anti-diagonal

$\hookrightarrow$ 6 matrices ✓

$J = \begin{bmatrix}1&0\\0&1\end{bmatrix} \Rightarrow tr=2,\ det=1 \Rightarrow \begin{bmatrix}1&\\&1\end{bmatrix}$ on diag, $\begin{bmatrix}&1\\0&\end{bmatrix}$ or $\begin{bmatrix}&0\\1&\end{bmatrix}$ or $\begin{bmatrix}&0\\0&\end{bmatrix}$ on antidiagonal

$\hookrightarrow$ 3 matrices

$J = \begin{bmatrix}0&0\\0&0\end{bmatrix} \Rightarrow tr=0,\ det=0$, 2 independent e-vecs

$\Rightarrow \begin{bmatrix}0&\\&0\end{bmatrix}$ on diag, $\begin{bmatrix}&0\\0&\end{bmatrix}$ on anti-diag

$\hookrightarrow$ 1 matrix

$J = \begin{bmatrix}0&1\\0&0\end{bmatrix} \Rightarrow tr=0,\ det=0$, 1 independent evec

$\Rightarrow \begin{bmatrix}0&\\&0\end{bmatrix}$ on diag, $\begin{bmatrix}&1\\0&\end{bmatrix}$ or $\begin{bmatrix}&0\\1&\end{bmatrix}$

$\hookrightarrow$ 2 matrices

$J = \begin{bmatrix}2&\\&0\end{bmatrix} \Rightarrow tr=2,\ det=0 \Rightarrow \begin{bmatrix}1&1\\1&1\end{bmatrix}$

$\hookrightarrow$ 1 matrix

$J = \begin{bmatrix}1&0\\0&-1\end{bmatrix} \Rightarrow tr=0,\ det=-1 \Rightarrow \begin{bmatrix}0&\\&0\end{bmatrix}$ on diag, $\begin{bmatrix}&1\\1&\end{bmatrix}$ on anti-diag,

$\hookrightarrow$ 1 ~~matrices~~ matrix

$J = \begin{bmatrix}1/2+\sqrt{5}/2&\\&1/2-\sqrt{5}/2\end{bmatrix} \Rightarrow tr=1,\ det=-1 \Rightarrow \begin{bmatrix}0&\\&1\end{bmatrix}$ or $\begin{bmatrix}1&\\&0\end{bmatrix}$ diag ✓

$\begin{bmatrix}&1\\1&\end{bmatrix}$ antidiag

$\hookrightarrow$ 2 matrices

6.6.17 a. T. They have different e-vals!, only singular matrices have $\lambda = 0$.

+

b. F. Every diagonalizable nonsymmetric matrix is similar to its diagonal eigenval matrix, which is symmetric.

c. F. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is similar to $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

d. T, $A+I$ has eigenvalues equal to those of A plus 1, so they are different.

6.6.18. AB has eigenvalues λ such that

1/2

$ABx = \lambda x$. Multiplying on the left by B,

$BABx = \lambda Bx$. Let $y = Bx$

$Bx \neq 0$?

$BAy = \lambda y$, which is the definition of an eigenval of BA. So BA and AB have the same eigenvals λ

6.6.20 +

a. If A and B are similar they have the same e-vals. Squaring them squares the eigenvalues, so A^2 and B^2 have the same eigenvals and are similar.

b. Consider A w/ $\lambda = x$. and B w/ $\lambda = -x$. The two do not have the same eigenvals. But A^2 and B^2 both have $\lambda = x^2$

c. Both have eigenvalues 3, 4 and 2 independent e-vectors

d. They have the same eigenvals 3, 3 but A has 2 independent e-vectors and B has only 1

e. This is $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{M^{-1}} \underbrace{\begin{bmatrix} A \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{M}$, so A is similar to the result, so they have the same e-vals

6.7.4 a. $A^TA = AA^T = A^2 = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$

+

$tr = 3$, $\det = 1 \Rightarrow \lambda = \frac{3}{2} \pm \frac{\sqrt{5}}{2}$

$\lambda_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}$, $\begin{bmatrix} 1/2 - \sqrt{5}/2 & 1 \\ 1 & -1/2 - \sqrt{5}/2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1/2 - \sqrt{5}/2 \end{bmatrix}}_{x_1} = 0$

$\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$ $\begin{bmatrix} 1/2 + \sqrt{5}/2 & 1 \\ 1 & -1/2 + \sqrt{5}/2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1/2 + \sqrt{5}/2 \end{bmatrix}}_{x_2} = 0$

After normalization, $\lambda = \frac{3}{2} + \frac{\sqrt{5}}{2}$, $x_1 = \begin{bmatrix} .525 \\ .850 \end{bmatrix}$

$\lambda_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}$, $x_2 = \begin{bmatrix} .850 \\ -.525 \end{bmatrix}$

b. $A = U\Sigma V^T = \begin{bmatrix} .850 & .525 \\ .525 & -.850 \end{bmatrix} \begin{bmatrix} 1.618 & 0 \\ 0 & .618 \end{bmatrix} \begin{bmatrix} .850 & .525 \\ -.525 & .850 \end{bmatrix}$

6.7.5 +

$Av_1 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} .850 \\ .525 \end{bmatrix} = \begin{bmatrix} 1.375 \\ .85 \end{bmatrix} = 1.618 \begin{bmatrix} .85 \\ .525 \end{bmatrix} = \sigma_1 u_1$

$Av_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} .525 \\ -.850 \end{bmatrix} = \begin{bmatrix} -0.325 \\ .85 \end{bmatrix} = .618 \begin{bmatrix} -.525 \\ .850 \end{bmatrix} = \sigma_2 u_2$

6.7.8 We have $Av_1 = u_1$, $Av_2 = u_2$, $Av_3 = u_3$ ...

+

This is $AV = U\Sigma$ for $\Sigma = I$. ?

So $A = U\Sigma V^T = UIV^T = \boxed{UV^T}$ where U and V are the matrixes made from the columns $u_1 - u_n$ and v_1 to v_n respectively.

6.7.14 3/4

Let $A = [u_1\ u_2] \begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix} [v_1\ v_2]^T$, then A can be made singular by setting σ_1 or σ_2 to 0.

?

6.7.16. $A+I$ doesn't just have a SVD w/ $\Sigma + I$ ... because the eigenvalues of $(A+I)^T(A+I)$ aren't simply one greater than those of A^TA

$\frac{1}{2}$

7.1.12 a. $T(2,2) = 2T(1,1) = (4,4)$

+

b. $T(3,1) = T(1,1) + T(2,0) = (2,2)$

c. $T(-1,1) = T(1,1) - T(2,0) = (2,2)$

d. $(a,b) = b(1,1) + \frac{a-b}{2}(2,0)$

$\Rightarrow T(a,b) = bT(1,1) + \left(\frac{a-b}{2}\right) T(2,0) = b(2,2)$

7.1.24 a. 1 eigenvalue of $A = 0 \Rightarrow \det A = 0$

+

b. both e-vals $> 0 \Rightarrow \det A > 0$

c. both e-vals of $A = 1 \Rightarrow \det A = 1$

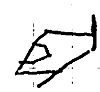
One side of the house consists of two points. These are two vectors that span $\mathbb{R}^2$. If each vector of these two is transformed to itself, since they form a basis for the input and output space and the ~~house~~ basis vectors are both unchanged, $A = I$.

7.1.30 $\begin{bmatrix} 1 & 0 \\ 0 & 0.1 \end{bmatrix}$ - stretched vertically down to 0.1 height.

+

$\begin{bmatrix} .5 & .5 \\ .5 & .5 \end{bmatrix}$ - flattened onto the line through $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the origin

$\begin{bmatrix} .5 & .5 \\ -.5 & .5 \end{bmatrix}$ - rotated 45° 

$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ - rotated 90° and skewed 

7.2.5. $T(v_1) = 0 + 1w_2 + 0$

4 $T(v_2) = 1w_1 + 0 + 1w_3$

$T(v_3) = 1w_1 + 0 + 1w_3$

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

7.2.6 a. $T(v_1 + v_2 + v_3) = 2w_1 + 2w_3 + 1w_2$

4 b. $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2\\1\\2\end{bmatrix}$

7.2.7 $T(v) = 0 \Rightarrow v = a(v_2 - v_3)$, The vectors

4 $a\begin{bmatrix}0\\1\\-1\end{bmatrix}$ are in the nullspace of A.

$T(v) = w_2$ for $v = v_1 + a(v_2 - v_3)$ the cosets?

7.2.15 $\frac{3}{4}$

a. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ b. $A^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

c. $\binom{2}{6} = 2\binom{1}{3}$, but $T\binom{2}{6} \neq 2T\binom{1}{3}$: ?

7.2.33 4 False. The vectors must be independent to provide a basis.

7.3.2 $(7,5,3,1) = 4(1,1,1,1) + 2(1,1,-1,-1)$ ~~+1(1,-1)~~

$+ 1(1,-1,0,0) + 1(0,0,1,-1)$

$\frac{1}{4}$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & -1 & 0 \\ 1 & -1 & 0 & 1 \\ 1 & -1 & 0 & -1 \end{bmatrix}\begin{bmatrix}4\\2\\1\\1\end{bmatrix}$$

?

[illegible]