R8

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18.06 PS4

Part 1: 12/10
Part 2: 13.75/16
Grade: 10

I. 4A. Let n be the dimensions of the matrix in question.

37

Then the following sequence of Matlab commands will give the dimension of P_{span}:

```
>> I = eye(n);                              (generate identity)
>> P = perms(1:n);                          (generate permutation vectors)
>> A = zeros(n^2, factorial(n));            (initialize A with the correct dimensions)
>> for x = 1:factorial(n)
>> t = I(:, P(x,:));                        (permute the identity)
>> A(:, x) = t(:)                           (convert the permutation matrix to a column and store it in A)
>> end
>> rank(A)                                  (find dimension of Pspan = rank of A)
```

The results are as follows (Matlab transcripts attached):

n	rank(A)
1	1
2	2
3	5
4	10
5	17

Thus it can be seen that an expression for the dimension of P_{span} is $(n-1)^2+1$.
This can be explained inductively: a 1x1 matrix obviously has only one independent permutation matrix. Next assume a nxn

matrix has $(n-1)^2+1$ independent permutation matrices. Now add another row and column, giving ~~a~~ $(n+1)\times(n+1)$ matrix. This adds new permutations that can be created by swapping the new row with any of the other rows; this creates n new permutation matrices. There are also $n-1$ possible permutations involving cycling the new row with 2 or more old rows. All other permutations are not independent. $(n-1)^2+1+n+(n-1) = (n^2-2n+1)+1+n+(n-1)$
$= n^2+1 = [(n+1)-1]^2+1$

(4B is on the next page)

2 The dimension of the row and column spaces is the rank r. It can be minimized by putting the 1s in the same row or same column, or by having two rows with 2 1s each in the same column, giving a row and column space with dimension 1.

The dimension of the nullspace is $n-r$, so it can be minimized by maximizing the rank by putting the 1s in different rows and columns. This gives $r=4 \Rightarrow$ dimension $= 6-4 = 2$

The four fundamental subspaces have dimensions $r, r, m-r, n-r$. The sum is $m+n$, so the placement of the 1s cannot affect the sum of dimensions.

4B. The projection matrix P can normally be found with $P = A(A^TA)^{-1}A^T$. However, for this matrix, A^TA is singular and thus cannot be inverted. We must replace the matrix A with another matrix of full column rank that has the same column space. This is the matrix

$B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ -2 & 1 \\ -1 & 0 \end{bmatrix}$, which can be computed with $B = \text{colbasis}(A)$.

3

Then $P = B\cdot(B^TB)^{-1}B$ is solvable!

$$\boxed{P = \begin{bmatrix} 1/2 & 0 & 0 & -1/2 \\ 0 & 1/2 & -1/2 & 0 \\ 0 & -1/2 & 1/2 & 0 \\ -1/2 & 0 & 0 & 1/2 \end{bmatrix}}$$

$PA = A$, as expected.
(Matlab transcript attached)

4D. 1. $E = (x-70)^2 + (x-80)^2 + (x-120)^2$

$\Rightarrow dE/dx = 2(x-70) + 2(x-80) + 2(x-120) = 0$

$\Rightarrow 6x - 540 = 0 \Rightarrow \boxed{x = 540/6 = 90}$

2

2. $\hat{x} = \dfrac{a^Tb}{a^Ta} = \dfrac{(1,1,1)\begin{pmatrix}70\\80\\120\end{pmatrix}}{(1,1,1)\begin{pmatrix}1\\1\\1\end{pmatrix}} = \dfrac{270}{3} = \boxed{90}$

$\hat{x}_{new} = \dfrac{(1,1,1,1)\begin{pmatrix}70\\80\\120\\130\end{pmatrix}}{(1,1,1,1)\begin{pmatrix}1\\1\\1\\1\end{pmatrix}} = \dfrac{400}{4} = 100$

$\hat{x}_{new} = \hat{x}_{old} + \frac{1}{4}(130 - \hat{x}_{old}) = 90 + \frac{1}{4}(130-90) = 90 + \frac{40}{4}$
$= 100$. These match, as expected.

II 3.5.19. a. might not
(+) b. are not
c. might be

(+) 3.5.23 a. T b. F Why?

3.5.25 a. False. A matrix may have multiple dependent columns but only one row.
(+) b. False consider $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$
c. True. They have dimension n.
d. False. They columns may be dependent.

3.5.29 2x3 matrices w/ columns that add to 0.
(1/2)

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}.$$

These matrices are obviously independent, because each has at least one non-zero position that is zero in every other matrix, and they span the space because the arbitrary matrix satisfying this property, $\begin{bmatrix} a & b & (-a-b) \\ c & d & (-c-d) \end{bmatrix}$ is a linear combination of these 4.

2x3 matrices w/ rows + cols that add to 0

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}.$$ This ~~is a~~ basis ~~by~~ the same reasoning as above.

one more is clear

3.6.4 a. $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ b. Does not exist. $r=1$, $n=3$, so the nullspace must have dimension 2
(+)

3.6.4 c. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ d. $\begin{bmatrix} -9 & -3 \\ 3 & 1 \end{bmatrix}$

e. Does not exist. The row space is the orthogonal complement of the nullspace and likewise for the colspace and left-nullspace, so if $C(A^T) = C(A)$, then $N(A) = N(A^T)$

3.6.8

	$C(A)$ + $C(A^T)$		$N(A)$	$N(A^T)$
A	3	3	2	0
B	3	3	3	2
C	0	0	2	3

E

3.6.9 a. row, null

b. column, left null

Consider $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$. The second row and second column are dependent on the first, so they have no pivots. All the pivots are in the upper-left A. This A is present in $[A]$, $\begin{bmatrix} A \\ A \end{bmatrix}$, and $\begin{bmatrix} A & A \\ A & A \end{bmatrix}$, so they have the same pivots and thus the same rank.

A

3.6.11 a. $r \le m$. $r \le n$.

b. $r < m$, so $m - r > 0$. The left nullspace has dimension of at least 1 so it must have some non-zero vectors in it.

A

3.6.21 a. u, w

b. v, z

c. If either u & w or v & z are dependent.

d. $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ rank(A) = 2

A

1/2

3.6.26. B. B. A. Why?

4.1.3 a. $\begin{bmatrix} 1 & 2 & -3 \\ 2 & -3 & 1 \\ -3 & 5 & -2 \end{bmatrix}$ b. Not possible. $\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix}$ is not $\perp$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

0+

c. $[0 \ 0]$

d. This means (1,1,1) is in the left nullspace; it cannot also be in the column space.

e. (1,1,1) cannot be in the nullspace and row space

4.1.7 $A^T A x$ must equal zero. This means x is in $N(A^T A) = N(A)$. So Ax must be zero. Ax is also in the column space because $C(A)$ must contain the zero column.

+

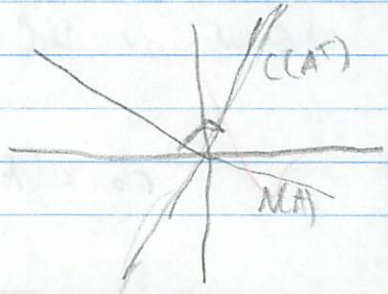
4.1.8 The row space is $C(A^T) = C(A)$. Thus the row and column spaces are the same, so they are both orthogonal to the other two spaces.

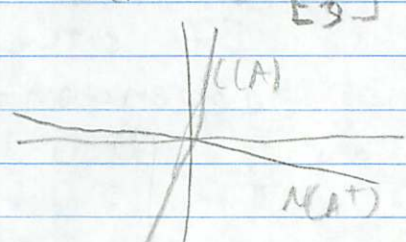
+

b. x is in the nullspace, z is in the column space. From above, these spaces are orthogonal, so $x \perp z$.

4.1.9 $R = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ $N(A) = c\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ $N(A^T)$ $d[3 \ -1]$

$C(A) = e\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $C(A^T) = f[1 \ 2]$



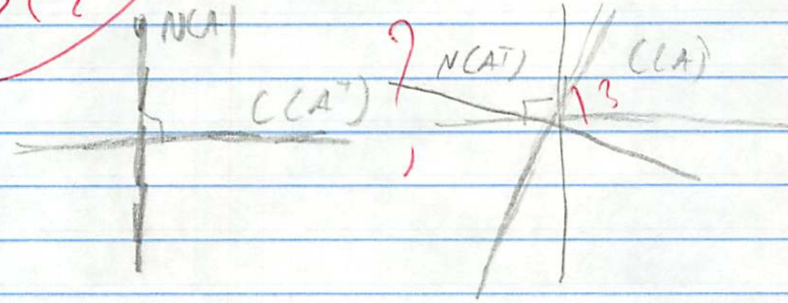


?

3/4

4.1.9 B: $R = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $N(B) = c\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $N(A^T) = \begin{bmatrix} -3, 1 \end{bmatrix}$

$C(A) = c\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $C(A^T) = \begin{bmatrix} 1 & 0 \end{bmatrix}$



1/2

4.1.24 a. The planes intersect. ✓ (1, 1, 0) in both

b. There are other parts of $\mathbb{R}^3$ not on either line Which?

c. Two non-orthogonal lines can still meet only at the origin?

+

4.1.26 $B = N(N(A)^T)$ This is a basis for the row space, because the row space ✓ is the orthogonal complement of A's nullspace, and B is necessarily orthogonal to $N(A)$.

```
>> n=3;
>> I=eye(n);
>> P=perms(1:n);
>> A=zeros(n^2,factorial(n));
>> for x=1:factorial(n)
t=I(:,P(x,:));
A(:,x)=t(:);
end
>> A

A =

     0     0     0     0     1     1
     0     0     1     1     0     0
     1     1     0     0     0     0
     0     1     0     1     0     0
     1     0     0     0     1     0
     0     0     1     0     0     1
     1     0     1     0     0     0
     0     1     0     0     0     1
     0     0     0     1     1     0

>> rank(A)

ans =

     5



>> n=4;
>> I=eye(n);
>> P=perms(1:n);
>> A=zeros(n^2,factorial(n));
>> for x=1:factorial(n)
t=I(:,P(x,:));
A(:,x)=t(:);
end
>> A

A =

  Columns 1 through 13

     0     0     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0     0     1
     0     0     0     0     0     0     1     1     1     1     1     1     0
     1     1     1     1     1     1     0     0     0     0     0     0     0
     0     0     0     0     1     1     0     0     0     0     1     1     0
     0     0     1     1     0     0     0     0     1     1     0     0     0
     1     1     0     0     0     0     0     0     0     0     0     0     1
     0     0     0     0     0     0     1     1     0     0     0     0     0
     0     1     0     1     0     0     0     1     0     1     0     0     0
     1     0     0     0     1     0     1     0     0     0     1     0     0
     0     0     1     0     0     1     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     1     0     0     1     1
```

```
     1     0     1     0     0     0     1     0     1     0     0     0     1
     0     1     0     0     0     1     0     1     0     0     0     1     0
     0     0     0     1     1     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     1     1     0     0

  Columns 14 through 24

     0     0     0     0     0     1     1     1     1     1     1
     1     1     1     1     1     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     0     0     0     0     0
     0     0     0     1     1     0     0     0     0     0     0
     0     0     0     0     0     0     0     1     1     0     0
     1     0     0     0     0     1     1     0     0     0     0
     0     1     1     0     0     0     0     0     0     1     1
     1     0     1     0     0     0     0     0     0     0     0
     0     0     0     0     0     1     0     0     0     1     0
     0     1     0     0     1     0     0     1     0     0     1
     0     0     0     1     0     0     1     0     1     0     0
     0     1     0     0     0     0     0     0     0     0     0
     0     0     0     0     0     0     1     0     0     0     1
     0     0     1     1     0     0     0     0     1     1     0
     1     0     0     0     1     1     0     1     0     0     0

>> rank(A)

ans =

    10



>> n=1

n =

     1

>> I=eye(n);
>> P=perms(1:n);
>> A=zeros(n^2,factorial(n));
>> for x=1:factorial(n)
t=I(:,P(x,:));
A(:,x)=t(:);
end
>> A

A =

     1

>> rank(A)

ans =
```

```
     1


>> n=2

n =

     2

>> I=eye(n);
>> P=perms(1:n);
>> A=zeros(n^2,factorial(n));
>> for x=1:factorial(n)
t=I(:,P(x,:));
A(:,x)=t(:);
end
>> A

A =

     0     1
     1     0
     1     0
     0     1

>> rank(A)

ans =

     2


>> n=5

n =

     5

>> I=eye(n);
>> P=perms(1:n);
>> A=zeros(n^2,factorial(n));
>> for x=1:factorial(n)
t=I(:,P(x,:));
A(:,x)=t(:);
end
>> rank(A)

ans =

    17
```

```
>> A=[1 0 -1 -1; 2 1 -1 -4; -2 -1 1 4; -1 0 1 1]

A =

     1     0    -1    -1
     2     1    -1    -4
    -2    -1     1     4
    -1     0     1     1

>> A*inv(A'*A)*A'
Warning: Matrix is singular to working precision.
(Type "warning off MATLAB:singularMatrix" to suppress this warning.)

ans =

   NaN   NaN   NaN   NaN
   NaN   NaN   NaN   NaN
   NaN   NaN   NaN   NaN
   NaN   NaN   NaN   NaN

>> B=colbasis(A)

B =

     1     0
     2     1
    -2    -1
    -1     0

>> P=B*inv(B'*B)*B'

P =

    0.5000         0         0   -0.5000
         0    0.5000   -0.5000         0
         0   -0.5000    0.5000         0
   -0.5000         0         0    0.5000

>> P*A

ans =

    1.0000         0   -1.0000   -1.0000
    2.0000    1.0000   -1.0000   -4.0000
   -2.0000   -1.0000    1.0000    4.0000
   -1.0000         0    1.0000    1.0000

>> P*A-A

ans =

   1.0e-15 *

    0.2220         0   -0.2220   -0.2220
         0         0         0         0
         0         0         0         0
```

```
   -0.2220         0     0.2220     0.2220
```