P1 : 10/10
P2 : 13/14
Grl : 10

R8

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18.06 PS7

I. 7A.

3

$$A = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 3 & 9 \\ 2 & 2 & 8 \end{bmatrix}$$ Consider the cofactors along the first

$$\begin{vmatrix} 3 & 9 \\ 2 & 8 \end{vmatrix} = 6, \quad -\begin{vmatrix} 2 & 9 \\ 2 & 8 \end{vmatrix} = 2, \quad \begin{vmatrix} 2 & 3 \\ 2 & 2 \end{vmatrix} = -2$$

This gives the vector $\boxed{\begin{pmatrix} 6 \\ 2 \\ -2 \end{pmatrix}}$ which is a basis for the nullspace

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$ Consider the cofactors along the second row

$$-\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1, \quad \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1, \quad -\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

This gives $\boxed{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$, a basis for $N(A)$.

7B.

3

$A = \begin{bmatrix} 2 & -c \\ -1 & 2 \end{bmatrix}$. $|A - \lambda I| = \lambda^2 - 4\lambda + 4 - c$

$$\lambda = \frac{4}{2} \pm \sqrt{4 - 4 - c} = 2 \pm \sqrt{-c}$$

a) 2 real e-vals : $c < 0$
b) 2 repeated e-vals : $c = 0$
c) 2 complex e-vals : $c > 0$

$$|A^TA| = |A^T||A| = |A|^2 = \boxed{(4-c)^2}$$

$$A^TA = \begin{bmatrix} 5 & -2c-2 \\ -2c-2 & 4+c^2 \end{bmatrix} \quad tr(A^TA) = 5 + 4 + c^2 = \boxed{9 + c^2}$$

We know that $tr(A^TA)$ and $\det(A^TA)$ must be positive, from above. Since the trace is the sum of the two eigenvalues and the determinant their product, and both are positive, we know that no eigenvalue can be negative.

7c. $\frac{da}{dt} = 0 \Rightarrow a(t) = a(0)$

4

$\frac{db}{dt} = a(t) \Rightarrow b(t) = a(0)t + b(0)$

$\frac{dc}{dt} = 2b(t) \Rightarrow c(t) = a(0)t^2 + 2b(0)t + c(0)$

$\frac{dz}{dt} = 3c(t) \Rightarrow z(t) = a(0)t^3 + 3b(0)t^2 + 3c(0)t + z(0)$

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 \end{bmatrix} \quad A^4 = \begin{bmatrix} 0 \end{bmatrix}$$

$A^n = 0$ for all $n \geq 4$, so all further terms are zero.

$$e^{At} = I + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ t & 0 & 0 & 0 \\ 0 & 2t & 0 & 0 \\ 0 & 0 & 3t & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t^2 & 0 & 0 & 0 \\ 0 & 3t^2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ t^3 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2 & 2t & 1 & 0 \\ t^3 & 3t^2 & 3t & 1 \end{bmatrix}$$

At $t=1$, $e^{A \cdot 1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix}$ as expected from Pascal.

1. $\frac{d}{dt} e^{At} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 2t & 2 & 0 & 0 \\ 3t^2 & 6t & 3 & 0 \end{bmatrix} \Bigg|_{t=0} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$

This equals A, as expected.

$(e^A)(e^A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 4 & 1 & 0 \\ 8 & 12 & 6 & 1 \end{bmatrix}$

$(e^{2A}) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2^2 & 2\cdot 2 & 1 & 0 \\ 2^3 & 3\cdot 2^2 & 3\cdot 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 4 & 4 & 1 & 0 \\ 8 & 12 & 6 & 1 \end{bmatrix}$ These are equal, so $(e^A)(e^A) = (e^{2A})$ as expected.

We know that $e^A e^A = e^{2A}$ is always true because it follows directly from the definition of e^{At}:

$e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$. This infinite series can be multiplied by itself to give:

$(e^A)(e^A) = \left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)\left(\sum_{i=0}^{\infty} \frac{A^i}{i!}\right)$.

The product of these infinite series is the another infinite series whose terms are the sum of the products of the terms from the original series with lower exponents:

$(e^A)(e^A) = \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{A^k}{k!} \frac{A^{(j-k)}}{(j-k)!} \right)$

$= \sum_{j=0}^{\infty} \left(\sum_{k=0}^{j} \frac{A^j}{j!} \binom{j}{k} \right)$

From elementary combinatorics we know that the sum of binomial coefficients has value:

$\sum_{k=0}^{j} \binom{j}{k} = 2^j$. Therefore:

$(e^A)(e^A) = \sum_{j=0}^{\infty} \frac{A^j 2^j}{j!} = \sum_{j=0}^{\infty} \frac{(A \cdot 2)^j}{j!} = e^{2A}$ by definition

II. § 3.6 A.

(+) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 0 \\ 0 & 4 & 1 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & -4 \\ 0 & 0 & 3 \end{bmatrix}$ $C^T = \begin{bmatrix} 3 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 3 \end{bmatrix}$

det A = 1·C $\Rightarrow A^{-1} = \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & -4/3 & 1 \end{bmatrix}$

b. $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ $C^T = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}$

det A = 2·3 − 1·2 + 0·1 = 4 $\Rightarrow A^{-1} = \begin{bmatrix} 3/4 & 2/4 & 1/4 \\ 2/4 & 1 & 2/4 \\ 1/4 & 2/4 & 3/4 \end{bmatrix}$

5.3.9 (3/4) Find A^{-1} by transposing the cofactor matrix. Then take the cofactors of the resulting matrix and transpose them to find A. (det A = det A^{-1} = 1, so no need to divide by the determinant)

5.3.21 (+) The determinant is the volume of the box bounded by the column vectors. It is largest when they are orthogonal and independent; the determinant is then $L_1 \cdot L_2 \cdot L_3$

b. The length of L_1, L_2, L_3 must be $\sqrt{3 \cdot 1} = \sqrt{3}$ So the max volume is $\sqrt{3}\sqrt{3}\sqrt{3} < 6$. not possible.

6.1.5 (+) $\begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 + 0 = 0 \Rightarrow \lambda = \{1, 1\}$ for A

$\begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix} = 0 \Rightarrow (1-\lambda)^2 - 0 = 0 \Rightarrow \lambda = \{1, 1\}$ for B

$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \Rightarrow (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = \{1, 3\}$ for A+B

The sum of the e-vals is not equal to the e-vals of the sum of the matrices.

6.1.10

+ A markov matrix has $\boxed{\lambda_1 = 1}$. tr $A = 1.4$,
so $\boxed{\lambda_2 = .4}$

$\begin{bmatrix} -.4 & .2 \\ .4 & -.2 \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 0$ $\quad \begin{bmatrix} .2 & .2 \\ .4 & .4 \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = 0$

$\hat{x}_1$ $\quad\quad \hat{x}_2$

For A^∞, $\boxed{\lambda_1 = 1}$ and $\boxed{\lambda_2 = 0}$ ($\text{tr } A^\infty = 1$)

$\begin{bmatrix} -2/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} = 0$, $\begin{bmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ -1 \end{bmatrix}} = 0$

$\hat{x}_1$ $\quad\quad \hat{x}_2$

For A_{100}, $\lambda_1 = 1^{100} = 1$ and $\lambda_2 = (.4)^{100} \approx 0$
so $A_{100} \approx A^\infty$

6.1.27. + A has rank 1, so the nullspace has 3 dimensions and there are 3 eigenvalues equal to zero. The fourth eigenvalue is 4, by inspection.

C has rank 2, so 2 eigenvals = 0
The other 2 eigenvals are $\lambda = 2$, for eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

6.1.32 a. + $Au = 0u \Rightarrow u$ is in the nullspace
The nullspace has dimension 1, because there is one 0 eigenval, so u is a basis.
$Av = 3v$ and $Aw = 5w$, so v and w are in the column space. The matrix has rank $3 - 1 = 2$, so the colspace has dimension 2. v and w are in the colspace and are independent, so they are a basis.

b. $Av = 3v \Rightarrow A \frac{v}{3} = v$

$Aw = 5w \Rightarrow A \frac{w}{5} = w$

By linearity, $v/3 + w/5$ is a particular solution. Add the nullspace for the general solution: $v/3 + w/5 + ut$ for all t. ✓

c. If $Ax = w$ were solvable, w would be in the column space. w is independent of v and u which are in the colspace so the colspace would have dimension 3. But we know it has dimension 2. ✓

6.2.5 (+)
a. False. ✓ A can have a $\lambda = 0 \Rightarrow \det A = 0$
b. True. ✓ $A = S\Lambda S^{-1}$
c. True ✓ S has full rank
d. False ✓ S may not have enough e-vectors

6.2.8 (3/4)

$$A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$= \begin{bmatrix} a+b & a-b \\ a-b & a+b \end{bmatrix} = \begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{b} & \tilde{a} \end{bmatrix}$$

6.2.10 (+)

$$A = \begin{bmatrix} 1/2 & 1/2 \\ 1 & 0 \end{bmatrix}$$

a $\begin{vmatrix} 1/2 - \lambda & 1/2 \\ 1 & -\lambda \end{vmatrix} = 0 \Rightarrow (1/2 - \lambda)(-\lambda) - 1/2 = 0$

$\lambda_1 = 1, \lambda_2 = -1/2$

$$\begin{bmatrix} -1/2 & 1/2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0 \qquad \begin{bmatrix} 1 & 1/2 \\ 1 & 1/2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 0$$

λ_1 λ_2

b. $A = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1/2 \end{bmatrix}\left(\frac{1}{-3}\begin{bmatrix} -2 & -1 \\ -1 & 1 \end{bmatrix}\right)$ ✓

$\Lambda^n \to \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ as $n \to \infty$

$A^n \to \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \frac{1}{3} \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$ as $n \to \infty$

2. $\begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = A^n \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} G_{n+1} \\ G_n \end{bmatrix} = \begin{bmatrix} 2/3 & -1/3 \\ 2/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} 2/3 \\ 2/3 \end{bmatrix}$ as $n \to \infty$

6.2.11

P a. T : A is not singular because 0 is not an eigenvalue

b. F : The matrix may have only one

c. F : independent eigenvector or multiple independent evectors. So we do not know if it is diagonalizable.

6.3.1 $\operatorname{tr} A = 5$, $\det A = 4$, so $\lambda_{1,2} = \{1, 4\}$

1/2 $\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0$ $\quad \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 0$

$x_1 \qquad x_2$

$u(0) = (5, -2) \to u = 2e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 3e^{-4t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

6.3.9 a. $\frac{d}{dt}(u_1^2 + u_2^2 + u_3^2) = 2u_1u_1' + 2u_2u_2' + 2u_3u_3'$

$= 2u_1(cu_2 - bu_3) + 2u_2(au_3 - cu_1) + 2u_3(bu_1 - au_2)$

$= 2cu_1u_2 - 2bu_1u_3 + 2au_2u_3 - 2cu_2u_1 + 2bu_1u_3 - 2au_2u_3$

$= \boxed{0}$

$\frac{d}{dt}\|u(t)^2\| = 0 \Rightarrow \boxed{\|u(t)\| \text{ is constant}}$

b. $\|e^{At}u(0)\| = \|u(0)\|$ requires e^{At} to be orthogonal. e^{At} is orthogonal when A is skew-symmetric

6.3.20 $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow e^A = I + A + A^2/2 + A^3/6 \ldots$

A

$A^n = A \Rightarrow e^A = I + A + A/2 + A/6 \ldots$

$\Rightarrow e^A = \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$

$B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \quad B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = B^n$ for $n \geq 2$

$e^B = I + B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$\boxed{\begin{aligned} e^A e^B &= \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & -1 \\ 0 & 1 \end{bmatrix} \\ e^B e^A &= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e-2 \\ 0 & 1 \end{bmatrix} \end{aligned}}$$

$A + B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow (A+B)^n = (A+B)$

$$\Rightarrow \boxed{\begin{aligned} e^{A+B} &= I + (A+B) + \frac{(A+B)}{2} + (A+B)/6 \\ &= \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}}$$

$$\boxed{e^A e^B \neq e^B e^A \neq e^{A+B}}$$