Part 1: 10/10
Part 2: 14/14
Grade: 10 Good

R8

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18 06 PS 8

5 $A_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \Rightarrow \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & +1 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow (2-\lambda)\left[(2-\lambda)^2 - 1\right] + 1\left[-(2-\lambda)\right]$

$\Rightarrow (2-\lambda)^3 - 2(2-\lambda) = 0 \Rightarrow (2-\lambda)\left((2-\lambda)^2 - 2\right) = 0$

$\Rightarrow \lambda = 2$ or $(2-\lambda)^2 = 2$

$\Rightarrow \boxed{\lambda_1 = 2-\sqrt{2},\ \lambda_2 = 2+\sqrt{2},\ \lambda_3 = 2}$

$\begin{bmatrix} \sqrt{2} & -1 & 0 \\ -1 & \sqrt{2} & -1 \\ 0 & -1 & \sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{bmatrix}}_{X_1} = 0$ ✓ $\begin{bmatrix} -\sqrt{2} & -1 & 0 \\ -1 & -\sqrt{2} & -1 \\ 0 & -1 & -\sqrt{2} \end{bmatrix} \underbrace{\begin{bmatrix} \sqrt{2} \\ -2 \\ \sqrt{2} \end{bmatrix}}_{X_2^T} = 0$ ✓

$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{X_3^T} = 0$ ✓ (by inspection)

$X_1 X_2^T = \begin{bmatrix} \sqrt{2} \\ 2 \\ \sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -2 & \sqrt{2} \end{bmatrix} = 2 - 4 + 2 = 0$

so the first two e-vecs X_1 and X_2 are orthog.

8b 5 $B_4 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$ We are given that $\lambda_1 = 2-\sqrt{2}$ and $\lambda_2 = 2+\sqrt{2}$ ✓

Using elimination, we find that $R_B = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.
B_4 has only 3 pivots, so it is rank 3 and has one zero eigenval: $\boxed{\lambda_3 = 0}$.

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \operatorname{tr} B_4 \Rightarrow 2+\sqrt{2} + 2 - \sqrt{2} + 0 + \lambda_4 = 6$ ✓

$\Rightarrow 4 + \lambda_4 = 6 \Rightarrow \boxed{\lambda_4 = 2}$

By inspection,

$$\underbrace{\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}}_{B} \cdot \underbrace{\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}}_{x_3} = 0 \qquad \underbrace{\begin{bmatrix} -1 & -1 & & \\ -1 & 0 & -1 & \\ & -1 & 0 & -1 \\ & & -1 & -1 \end{bmatrix}}_{B-2I} \underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{x_4} = 0$$

$$x_3 x_4^T = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 1 \end{bmatrix} = 1 - 1 - 1 + 1 = 0$$

so $x_3 \perp x_4$

B. Define a matrix X whose columns are the vectors $x_1, x_2, x_3 \ldots x_n$ that we want to test $x_i^T A x_i \leq 0$ with. Then we can perform all the $x_i^T A x_i$ multiplications simultaneously by taking the diagonal of the product $X^T A X$

For $A = \text{pascal}(6)$, all $x^T A x > 0$. The matrix is positive definite. eigenvalues are all positive.

$A = \text{ones}(6)$ is not positive definite. Five eigenvalues are zero. $(1\,1\,1\,1\,1\,1)\, A \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} = 0$

$A = \text{hilb}(6)$ is also only positive semidefinite. One of the eigenvalues is zero. None of the vectors tested give $x^T A x \leq 0$, but there is some vector that gives $x^T A x = 0$

$A = \text{ones}(6) - \text{eye}(6)$ is not positive definite. Five of the eigenvalues are negative. many vectors give $x^T A x < 0$.

Matlab transcripts attached.

II. 6.4.5

(+) $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & -2 \\ 2 & -2 & -\lambda \end{vmatrix}$ $(1-\lambda)[(-1-\lambda)(-\lambda)-4]+2[-2(-1-\lambda)]=0$

$\Rightarrow \lambda_1 = 0,\ \lambda_2 = 3,\ \lambda_3 = -3$

$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & -2 \\ 2 & -2 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}}_{x_1} = 0 \quad \begin{bmatrix} -2 & 0 & 2 \\ 0 & -4 & -2 \\ 2 & -2 & -3 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1/2 \\ 1 \end{bmatrix}}_{x_2} = 0$

$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 2 & -2 \\ 0 & 0 & 3 \end{bmatrix} \underbrace{\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}}_{x_3}$

Thus $A = Q \Lambda Q^T$ for $Q = \begin{bmatrix} -2/3 & 2/3 & -1/3 \\ -2/3 & -1/3 & 2/3 \\ 1/3 & 2/3 & 2/3 \end{bmatrix}$

6.4.13 (+) x is not necessarily real, so neither is $\lambda = \frac{x^T A x}{x^T x}$

6.4.11 at end

6.4.11 $\begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$ tr = 6, det = 8 $\Rightarrow \lambda_1 = 4,\ \lambda_2 = 2$

$x_1 = \binom{1}{1},\ x_2 = \binom{1}{-1}$

$A = Q \Lambda Q^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

$= 4 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$

6.4.13. (+) The matrix is skew-symmetric and orthogonal. The only pure imaginary numbers w/ length 1 are $\pm i$, so those must be its eigenvals. The trace is 0, so the eigenvals must be $\boxed{i, i, -i, -i}$ to add to 0.

6.4.22. (+) A skew-symmetric matrix has $A^T = -A$ so $A^T A = -AA = A \cdot -A = AA^T$

$\Rightarrow$ it is normal

An orthogonal matrix has $Q^TQ = I = QQ^T$, so it is normal.

For $\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$, $AA^T = A^TA \Rightarrow \begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}\begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix} = \begin{bmatrix} a & -1 \\ 1 & d \end{bmatrix}\begin{bmatrix} a & 1 \\ -1 & d \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} a^2+1 & -a+d \\ -a+d & 1+d^2 \end{bmatrix} = \begin{bmatrix} a^2+1 & a-d \\ a-d & 1+d^2 \end{bmatrix}$$

This requires that $\boxed{a = d}$.

6.4.25 $A = Q\Lambda Q^T$ possible $\Rightarrow$ A symmetric.

This requires $\boxed{b = 1}$

$A = S\Lambda S^{-1}$ impossible $\Rightarrow$ not enough e-vecs

This requires at least one e-val 0, so the determinant must be 0; $\boxed{b = 0}$

6.5.1 det $A_1 = -1$, so not pos def.

A_2's first upper left det is negative. So it is not pos definite.

A_3's determinant is 0, so not pos def.

A_4 has upper-left det = 1 and det $A_4 = 1$. It is positive definite; $\boxed{A_4 \text{ only}}$

6.5.4 $f(x,y) = \begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. The matrix is not positive definite (its determinant is -1), so x^TAx does not have a min at $\vec{x} = 0$.

$f(x,y) = (x+2y)^2 - y^2$

$f(2,-1)$ is negative.

6.5.7 $x^TA^TAx = (Ax)^T(Ax)$. This can equal zero only when $Ax = 0$. The nullspace of A is only zero, so $x^TA^TAx = 0$ only if $x = 0$. Otherwise $(Ax)^T(Ax) = |Ax|^2$, which is always positive, so

The matrix is positive definite.

6.5.4. $A = \begin{bmatrix} 4 & -4 & 8 \\ -4 & 4 & -8 \\ 8 & -8 & 16 \end{bmatrix}$ by expanding $4(x_1 - x_2 + 2x_3)^2$

A has rank 1, elimination gives $\begin{bmatrix} 4 & -4 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

The only pivot is 4. The matrix is singular, so its determinant is zero. Two of the eigenvalues are zero because the rank is 1, so the third is equal to tr A = 24

$\boxed{\lambda_1 = \lambda_2 = 0, \lambda_3 = 24}$

6.5.12 $A = \begin{bmatrix} c & 1 & 1 \\ 1 & c & 1 \\ 1 & 1 & c \end{bmatrix}$ To be positive definite, we must have $c > 0$, $c^2 - 1 > 0$, and $c^3 - 3c + 2 > 0$

$c^2 > 1 \Rightarrow c > 1$ (or $c < -1$, but this is extraneous)

$c^3 - 3c + 2 = c(c^2 - 3) - 2 > 0$. This is always true if $c > 1$. So the matrix is positive definite if $\boxed{c > 1}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & d & 4 \\ 3 & 4 & 5 \end{bmatrix}$. We need $1 > 0$, $d - 4 > 0$, $-4d + 12 > 0$

$d > 4$, $12 > 4d \Rightarrow d < 3$

so $3 < d < 4$ must hold.

6.5.15. $x^T(A+B)x = x^TAx + x^TBx$ by linearity.

We know that x^TAx and x^TBx are > 0 because A and B are pos. def. So their sum must be positive: $x^T(A+B)x > 0$. This means A + B must be positive definite.

6.5.20. a. If it were singular, $\det A = 0$, which would mean A is not positive definite. So all positive definite matrices are invertible.

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b. A permutation matrix has only one non-zero entry per column. Unless all of these are on the diagonal ($P = I$), then at least one of the pivots is zero.

c. A projection matrix has eigenvalues all equal to 0 ~~or~~ 1. The only projection matrix with all eigenvalues 1 is I; all others have some eigenvalues $= 0$. A positive definite matrix cannot have any eigenvalue equal to 0.

d. The pivots of a diagonal matrix are on the diagonal. If they are all positive, the matrix is positive definite.

e. Some of the upper-left smaller determinants might be ≤ 0. Consider for example, $\begin{bmatrix} -1 & -2 \\ 2 & 1 \end{bmatrix}$

6.5.22. $A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$ $\lambda_1 + \lambda_2 = 10$, $\lambda_1 \lambda_2 = 9$

1

$\Rightarrow \lambda_1 = 1, \lambda_2 = 9$

$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x_1} = 0$, $\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{x_2} = 0$

$A = Q \Lambda Q^T = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

$$\Rightarrow A^{1/2} = Q\Lambda^{1/2}Q^T = \frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 3\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

$$\boxed{= \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}}$$

$$A^{1/2}\cdot A^{1/2} = \begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix} = \begin{bmatrix}5 & -4\\ -4 & 5\end{bmatrix} = A$$

$$A = \begin{bmatrix}10 & 6\\ 6 & 10\end{bmatrix} \qquad \lambda_1 + \lambda_2 = 20, \quad \lambda_1\lambda_2 = 64$$

$$\lambda_1 = 4, \quad \lambda_2 = 16$$

$$\begin{bmatrix}6 & 6\\ 6 & 6\end{bmatrix}\underbrace{\begin{bmatrix}1\\ -1\end{bmatrix}}_{x_1} = 0 \qquad \begin{bmatrix}-6 & 6\\ 6 & -6\end{bmatrix}\underbrace{\begin{bmatrix}1\\ 1\end{bmatrix}}_{x_2} = 0$$

$$A = Q\Lambda Q^T = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 16\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$A^{1/2} = Q\Lambda^{1/2}Q^T = \frac{1}{2}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}2 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

$$= \boxed{\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}}$$

$$A^{1/2}\cdot A^{1/2} = \begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix}\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} = \begin{bmatrix}10 & 6\\ 6 & 10\end{bmatrix} = A$$

6.4.11

$$\begin{bmatrix}3 & 1\\ 1 & 3\end{bmatrix} \qquad tr = 6,\ det = 8 \Rightarrow \lambda_1 = 4,\ \lambda_2 = 2$$

$$x_1 = \begin{pmatrix}1\\ 1\end{pmatrix}, \quad x_2 = \begin{pmatrix}1\\ -1\end{pmatrix}$$

$$A = Q\Lambda Q^T = \begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}4 & 0\\ 0 & 2\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\\ 1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}$$

$$\boxed{A = 4\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix} + 2\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\end{bmatrix}}$$

$$B = \begin{bmatrix}9 & 12\\ 12 & 16\end{bmatrix} \qquad tr = 25,\ det = 0 \Rightarrow \lambda_1 = 0,\ \lambda_2 = 25$$

$$x_1 = \begin{pmatrix}-4/5\\ 3/5\end{pmatrix}, \quad x_2 = \begin{pmatrix}3/5\\ 4/5\end{pmatrix}$$

$$B = Q\Lambda Q^T = \begin{bmatrix}-4/5 & 3/5\\ 3/5 & 4/5\end{bmatrix}\begin{bmatrix}0 & 0\\ 0 & 25\end{bmatrix}\begin{bmatrix}-4/5 & 3/5\\ 3/5 & 4/5\end{bmatrix}$$

$$\boxed{B = 25\begin{bmatrix}3/5\\ 4/5\end{bmatrix}\begin{bmatrix}3/5 & 4/5\end{bmatrix}}$$

```
>> x=ones(6,12);

>> x(:,2)=[1;-1;1;-1;1;-1];

>> x(:,3:12)=randn(6,10)

x =

  Columns 1 through 9

    1.0000    1.0000   -0.2086    1.8705   -0.9640   -0.1170    0.2291   -0.2361
0.6518
    1.0000   -1.0000    0.7559   -1.2090   -2.3792    0.1685   -0.9595   -0.0755   -
0.3771
    1.0000    1.0000    0.3757   -0.7826   -0.8382   -0.5012   -0.1460   -0.3586   -
0.6614
    1.0000   -1.0000   -1.3454   -0.7673    0.2573   -0.7051    0.7445   -2.0776
0.2490
    1.0000    1.0000    1.4819   -0.1072   -0.1838    0.5082   -0.8905   -0.1435   -
0.3835
    1.0000   -1.0000    0.0327   -0.9771   -0.1676   -0.4209    0.1391    1.3933   -
0.5285

  Columns 10 through 12

    0.0554   -2.4240   -1.1264
    1.2538   -0.2238   -0.8150
   -2.5200    0.0581    0.3666
    0.5849   -0.4246   -0.5861
   -1.0081   -0.2029    1.5374
    0.9443   -1.5131    0.1401

>> A=pascal(6)

A =

     1     1     1     1     1     1
     1     2     3     4     5     6
     1     3     6    10    15    21
     1     4    10    20    35    56
     1     5    15    35    70   126
     1     6    21    56   126   252

>> diag(x'*A*x)

ans =

  923.0000
  111.0000
   71.5874
  440.1998
   55.6777
   35.5595
    9.9028
  220.4151
  136.2711
   59.8160
  756.6184
  164.1175
```

```
>> eig(A)
```



```
ans =

    0.0030
    0.0643
    0.4893
    2.0436
   15.5535
  332.8463

>> A=ones(6)

A =

     1     1     1     1     1     1
     1     1     1     1     1     1
     1     1     1     1     1     1
     1     1     1     1     1     1
     1     1     1     1     1     1
     1     1     1     1     1     1

>> diag(x'*A*x)

ans =

   36.0000
         0
    1.1929
    3.8916
   18.2795
    1.1396
    0.7802
    2.2440
    1.1021
    0.4758
   22.3760
    0.2336

>> eig(A)

ans =

   -0.0000
         0
         0
         0
    0.0000
    6.0000

>> A=hilb(6)

A =

    1.0000    0.5000    0.3333    0.2500    0.2000    0.1667
    0.5000    0.3333    0.2500    0.2000    0.1667    0.1429
```

```
    0.3333    0.2500    0.2000    0.1667    0.1429    0.1250
```



```
    0.2500    0.2000    0.1667    0.1429    0.1250    0.1111
    0.2000    0.1667    0.1429    0.1250    0.1111    0.1000
    0.1667    0.1429    0.1250    0.1111    0.1000    0.0909

>> diag(x'*A*x)

ans =

    7.8385
    0.4385
    0.1801
    1.4845
    6.8092
    0.1924
    0.1642
    0.6815
    0.2282
    0.0828
    8.8496
    1.5512

>> eig(A)

ans =

    0.0000
    0.0000
    0.0006
    0.0163
    0.2424
    1.6189

>> A=ones(6)-eye(6)

A =

     0     1     1     1     1     1
     1     0     1     1     1     1
     1     1     0     1     1     1
     1     1     1     0     1     1
     1     1     1     1     0     1
     1     1     1     1     1     0

>> diag(x'*A*x)

ans =

   30.0000
   -6.0000
   -3.5704
   -3.2360
   10.8591
   -0.0862
   -1.5809
   -4.2246
```

```
    -0.3908
```



```
    -9.6996
    13.9361
    -4.5604

>> eig(A)

ans =

    -1.0000
    -1.0000
    -1.0000
    -1.0000
    -1.0000
     5.0000
```