18 very good, not typed though.

Dan Ports R6
18.06 PS9

The eigenvalues and eigenvectors of a matrix A are the values λ and vectors x such that $Ax = \lambda x$. To find these graphically, we find the vectors x such that Ax is parallel to λx. Since x is restricted to unit-length vectors, the length of Ax will be the eigenvalue associated with eigenvector x. This is demonstrated for 3 matrices, one with 2 eigenvalues and eigenvectors, one with only one real eigenvector, and one with no real eigenvectors.

The SVD finds an orthonormal basis U for the column space and an orthonormal basis V for the row space that can be related by $AV = U\Sigma$. The demonstration finds these values by selecting an orthonormal matrix V whose transformation AV leads to an orthonormal matrix $U\Sigma$. We know that SVD always exists because A^TA is positive semidefinite, so it is diagonalizable. If A is singular, then some of its singular values σ will be zero. In this case, the remaining columns of U and V not found from diagonalizing A^TA will be filled with orthonormal bases for the left nullspace and nullspace respectively. The singular values σ are either the square roots of the eigenvalues of A^TA or AA^T or they are zero.

The possible values of x always form a circle because they are restricted to be unit vectors. Ax forms an ellipse:

$|Ax|^2 = (Ax)^T(Ax) = x^T A^T A x$. $A^T A$ is always diagonalizable, so $x^T A^T A x$ can be written as $x^T Q \Delta Q^T x$. This can be written as a difference of squares with coefficients set by the eigenvalues of $A^T A$. This translates to an ellipse with axis lengths $\sqrt{\lambda_1}$ and $\sqrt{\lambda_2}$ where λ_1 and λ_2 are eigenvalues of $A^T A$. These are the singular values σ_1 and σ_2 of A. Their associated directions are the eigenvectors of $A^T A$.

So the major axis of the ellipse has length σ_x, where σ_x is the largest singular value, and direction v_x where v_x is the associated vector in V.

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

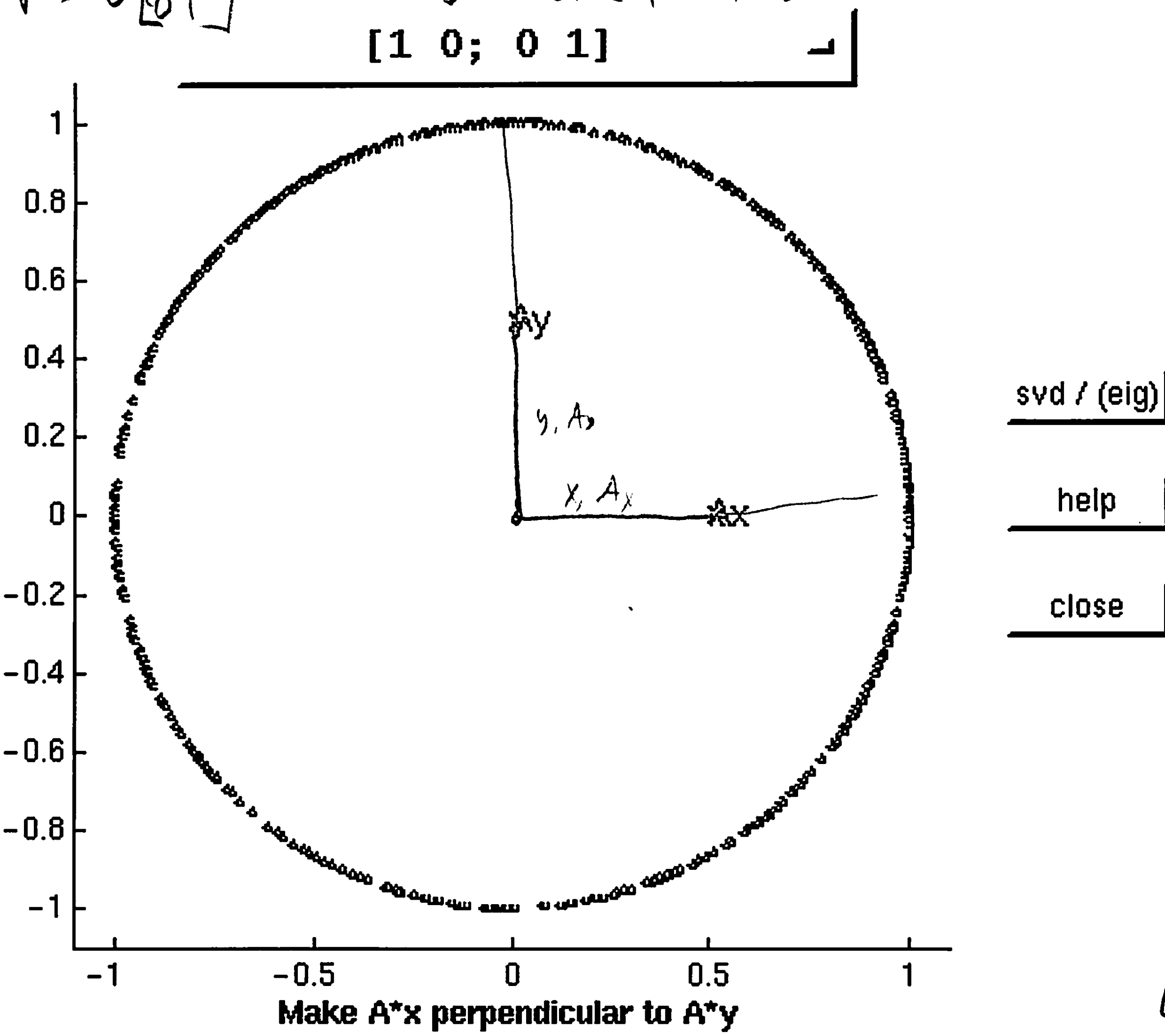
$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \sigma_1 = \sigma_2 = 1$

ATA has infinite possible eigenvectors. For purposes of diagonalization, choose an orthogonal pair. For example, $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ but any other orthonormal ones work too.

$A^TA = V \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V^T$

Then $AV = U\Sigma$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V = U \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ So U must equal V



For the SVD, we can thus choose

$A = U \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V^T$ for any orthogonal $U = V$,

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, but there are infinitely many other possibilities.

$A = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

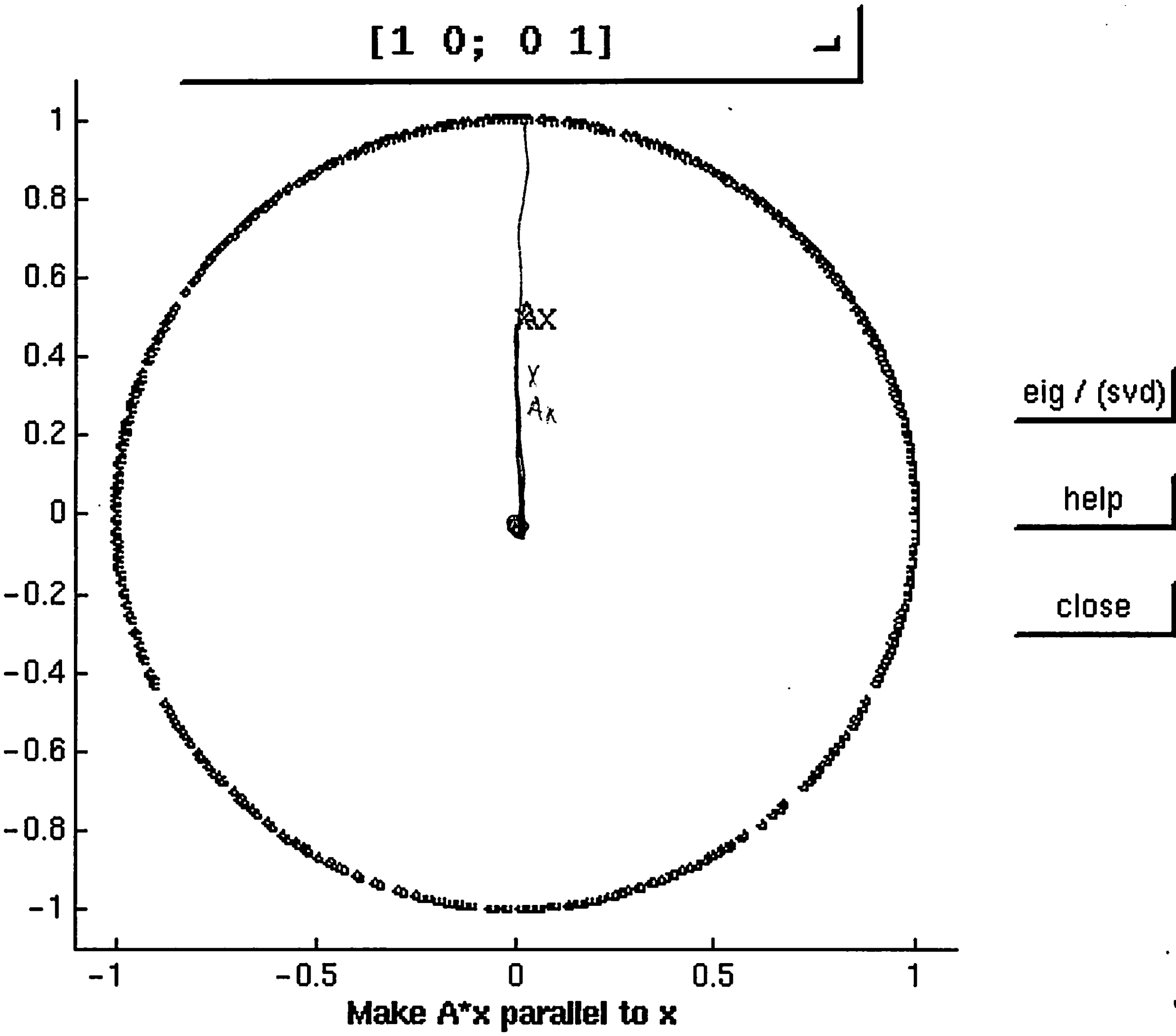
$A-\lambda I = \begin{bmatrix}1-\lambda&0\\0&1-\lambda\end{bmatrix}$

$(1-\lambda)(1-\lambda)=0$

$\Rightarrow \boxed{\lambda_1=\lambda_2=1}$

$\begin{bmatrix}0&0\\0&0\end{bmatrix}\begin{bmatrix}x\end{bmatrix}=0$

Any vector x is an eigenvector.



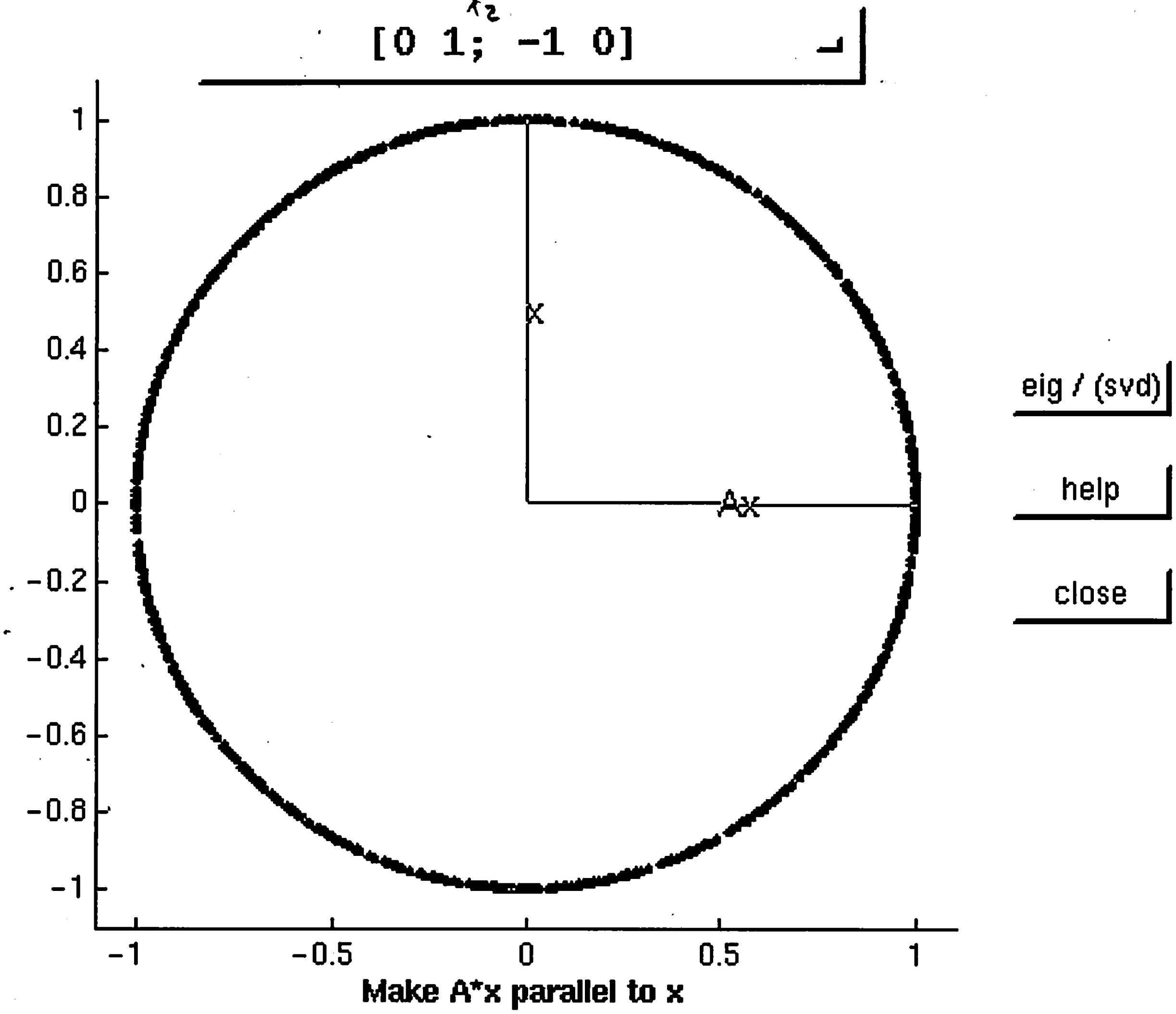
$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

A is a rotation matrix. It has no real eigenvalues! $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 + 1 = 0$

$\Rightarrow \lambda = \pm i$

The eigenvectors are $\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \underset{X_1}{\begin{bmatrix} 1 \\ i \end{bmatrix}} = 0$

$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \underset{\lambda_2}{\begin{bmatrix} 1 \\ -i \end{bmatrix}} = 0$

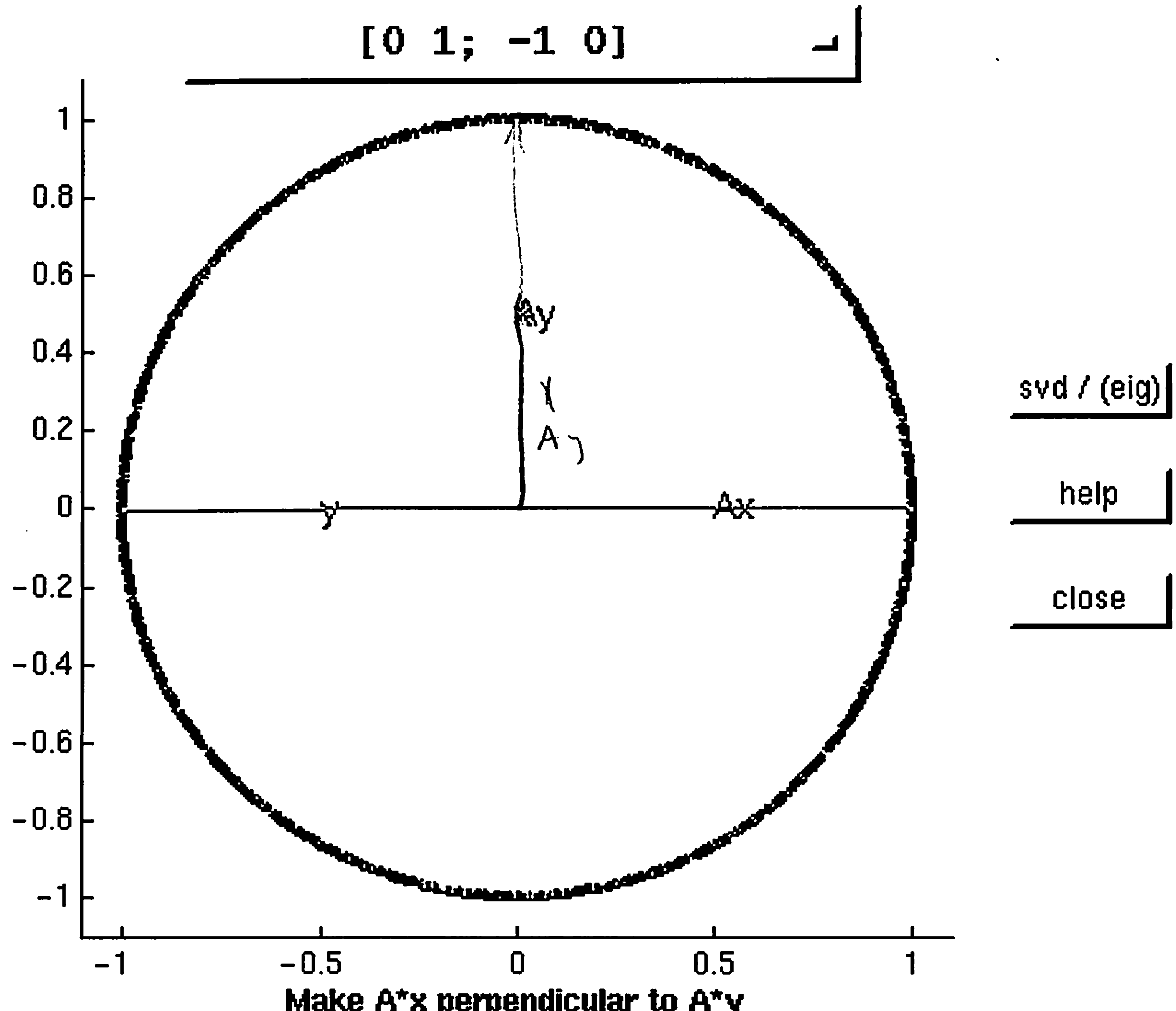


Trying all possible values for $\vec{x}$ on the unit circle reveals that $A\vec{x}$ is always perpendicular to $\vec{x}$, never parallel.

$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ | $A^T A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $A^T A$ has repeated eigenvalue 1 and infinite possible eigenvectors.

So in the diagonalization $A^T A = V \Sigma^T \Sigma V^T$, $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and V is any orthonormal pair of vectors. (see p. 2)

$AV = U\Sigma \Rightarrow \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} V = U \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Therefore, $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} V$



So $A = U\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} V^T$ for any orthogonal 2x2 matrix V and $U = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} V$. For example,

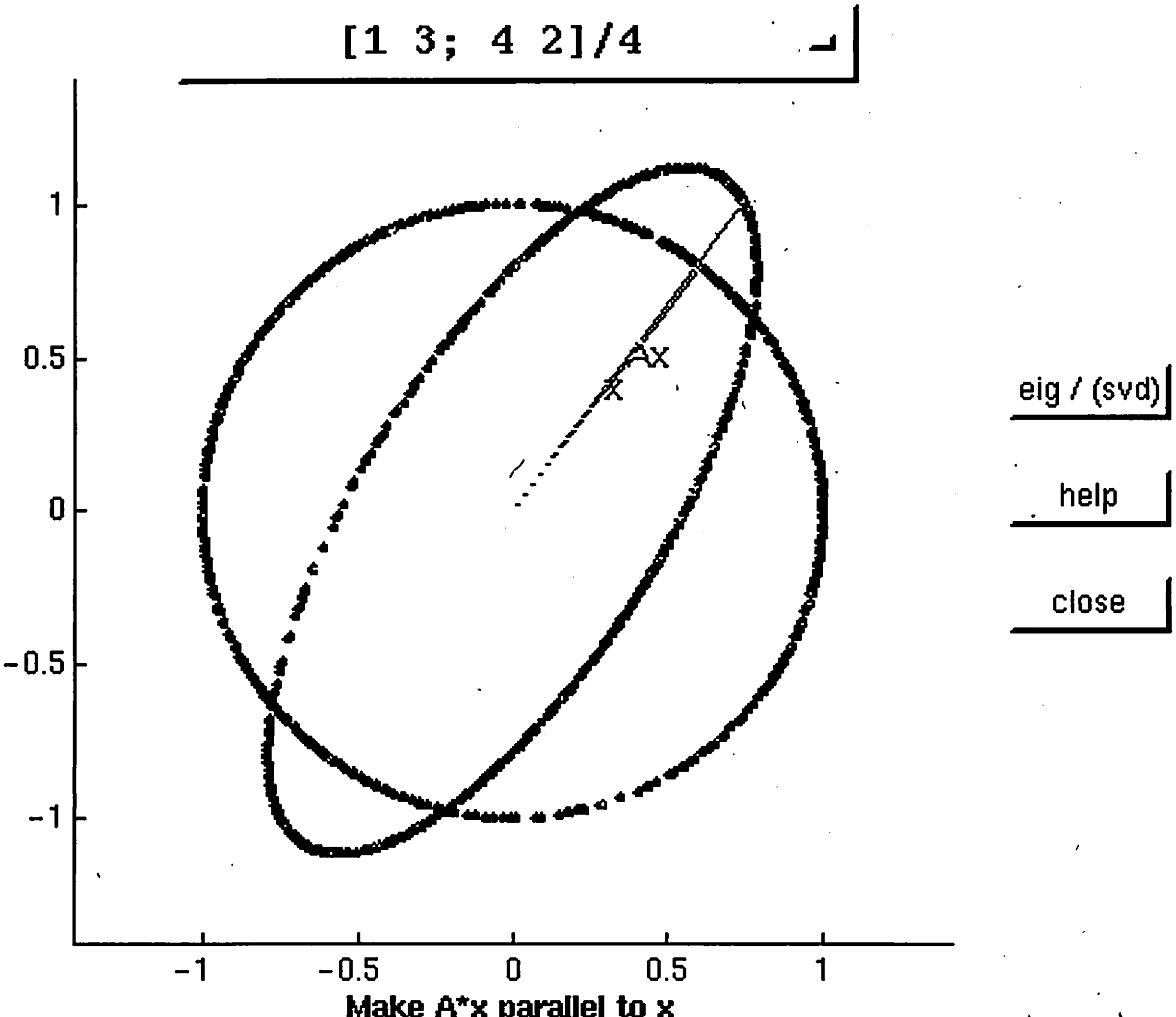
$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } A = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

or any of an infinite possible decompositions.

$$A = \boxed{\begin{bmatrix} 1/4 & 3/4 \\ 1 & 2/4 \end{bmatrix}}$$

$$\lambda_1 + \lambda_2 = \operatorname{tr} A = 3/4$$
$$\lambda_1 \cdot \lambda_2 = \det A = 1/8 - 3/4 = -5/8$$
$$\Rightarrow \lambda_1 = -1/2, \quad \lambda_2 = 5/4$$

$$\begin{bmatrix} 3/4 & 3/4 \\ 1 & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{x_1} = 0 \qquad \begin{bmatrix} -1 & 3/4 \\ 1 & -3/4 \end{bmatrix} \underbrace{\begin{bmatrix} 1 \\ 3/4 \end{bmatrix}}_{x_2} = 0$$



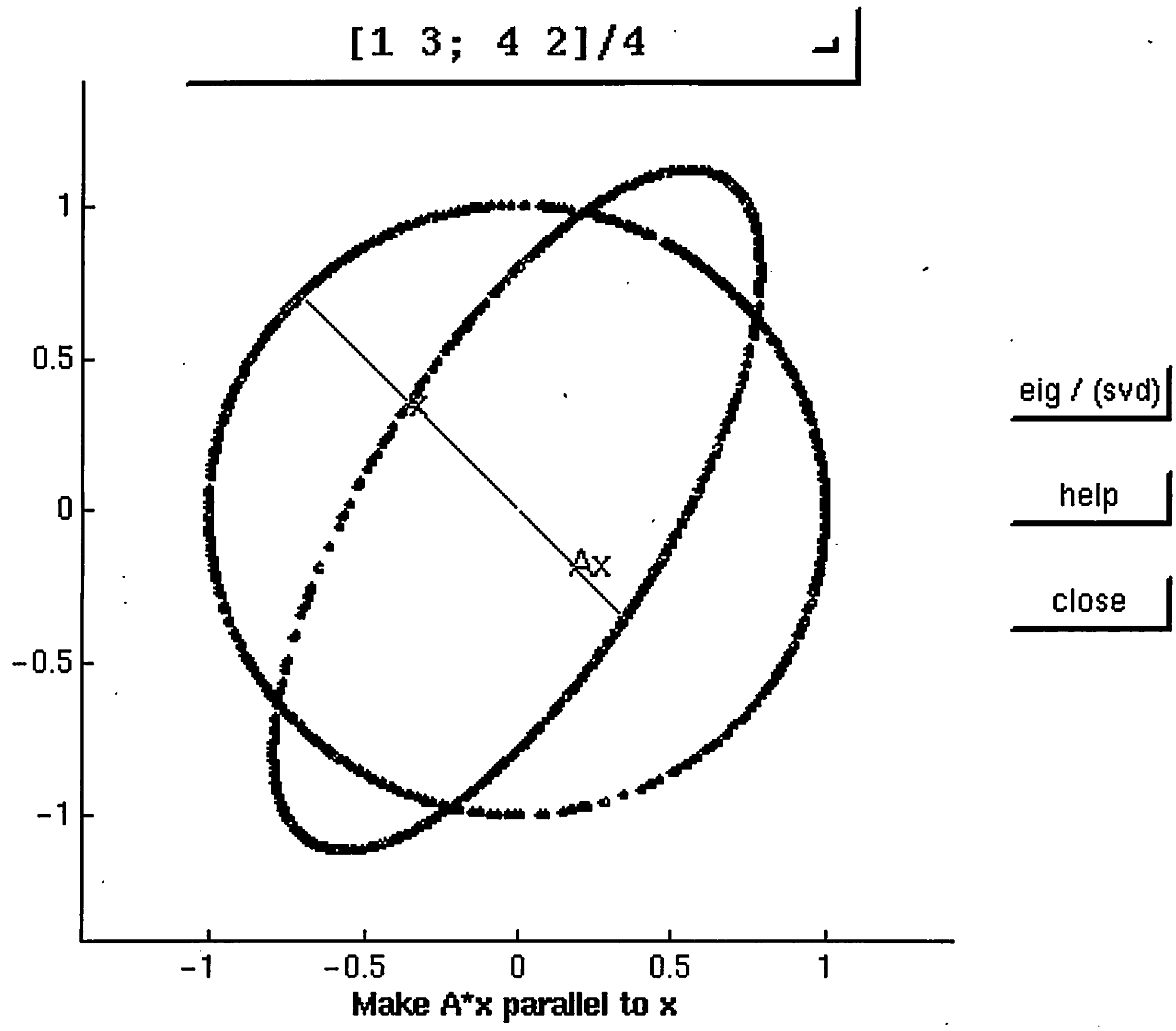
This picture indicates $A x_2 = \lambda_2 x_2$ ∴ $x_2 = \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix}$ (from above, normalized)

$$\lambda_2 = 5/4 \Rightarrow A x_2 = \lambda_2 x_2 = \begin{bmatrix} 1 \\ 3/4 \end{bmatrix}$$

$A = \begin{bmatrix} 1/4 & 3/4 \\ 1 & 1/2 \end{bmatrix}$

This is the 2nd eigenvector. From the previous page we have

$\lambda_1 = -1/2$, $x_1 = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ (normalized).



The vector shown is x_1. We see that $A x_1 = \lambda_1 x_1$; $A \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -1/2 \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

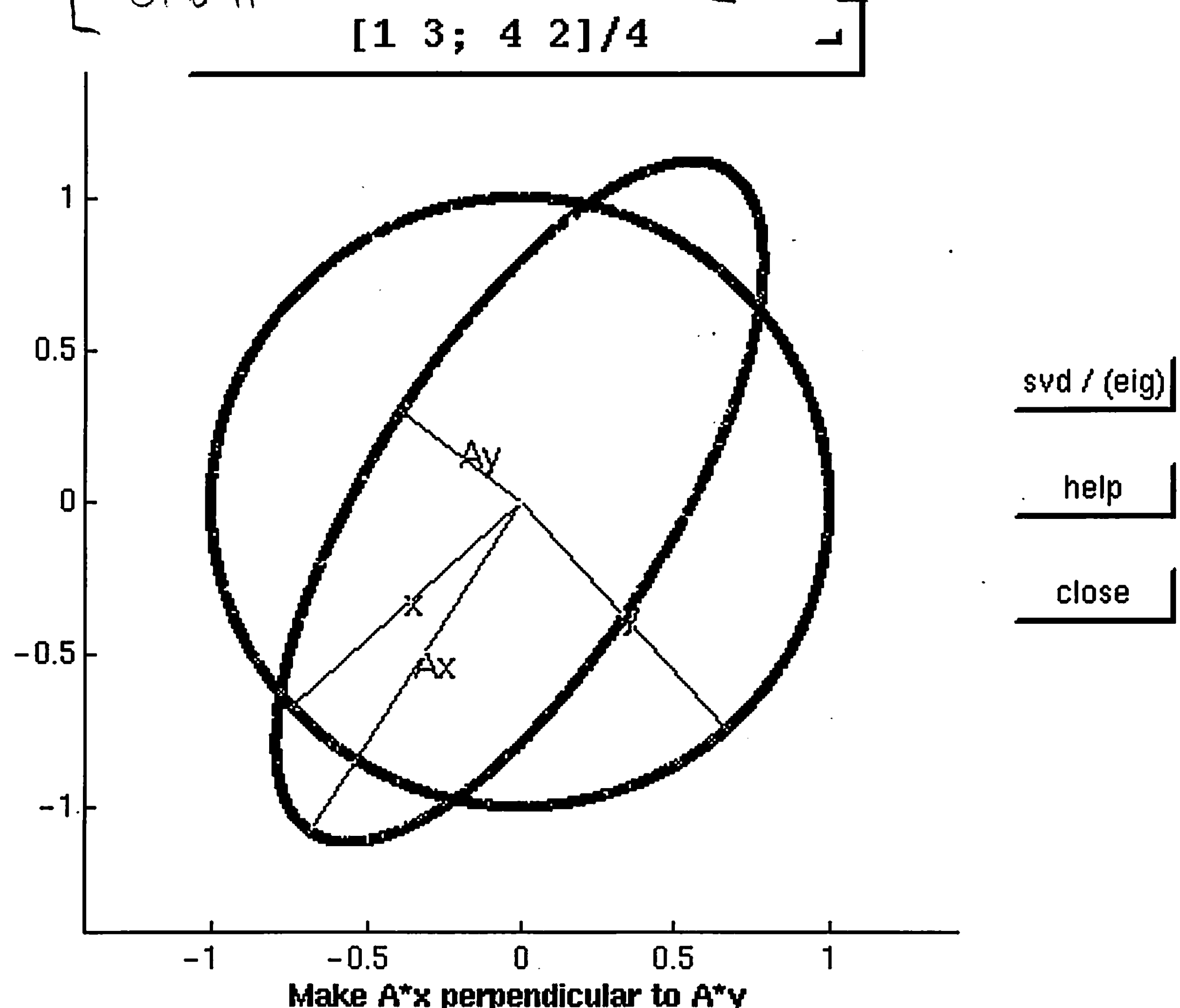
$$A = \begin{bmatrix} 1/4 & 3/4 \\ 1 & 1/2 \end{bmatrix} \qquad A^T A = \begin{bmatrix} 17/16 & 11/16 \\ 11/16 & 13/16 \end{bmatrix} \qquad \begin{array}{l} \lambda_1 + \lambda_2 = \operatorname{tr} A = \frac{30}{16} \\ \lambda_1 \lambda_2 = \det A = \frac{25}{64} \end{array}$$

$$\lambda_1 = 1.636 \qquad \lambda_2 = 0.239$$

$$x_1 = \begin{bmatrix} -0.767 \\ -0.641 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0.641 \\ -0.767 \end{bmatrix}$$ from Matlab

Therefore $A^T A$ diagonalizes as $V \Sigma^T \Sigma V^T$ for

$$V = \begin{bmatrix} -0.767 & 0.641 \\ -0.641 & -0.767 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} \sqrt{1.636} & 0 \\ 0 & \sqrt{0.239} \end{bmatrix}$$



$$AV = U\Sigma \Rightarrow \begin{bmatrix} 1/4 & 3/4 \\ 1 & 1/2 \end{bmatrix} \begin{bmatrix} -0.767 & 0.641 \\ -0.641 & -0.767 \end{bmatrix} = U \begin{bmatrix} 1.279 & 0 \\ 0 & 0.489 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -0.673 & -0.416 \\ -1.088 & -0.257 \end{bmatrix} = U \begin{bmatrix} 1.279 & 0 \\ 0 & 0.489 \end{bmatrix}$$

$$U = \begin{bmatrix} -0.526 & -0.851 \\ -0.851 & 0.526 \end{bmatrix}$$

$$A = U\Sigma V^T = \begin{bmatrix} -0.526 & -0.851 \\ -0.851 & 0.526 \end{bmatrix} \begin{bmatrix} 1.279 & 0 \\ 0 & 0.489 \end{bmatrix} \begin{bmatrix} -0.767 & 0.641 \\ -0.641 & -0.767 \end{bmatrix}$$

x and y are shown as the 2nd and 1st cols of V^T and Ax and Ay are the 2nd and 1st cols of $U\Sigma$