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1 36
2 30
3 24

(90)

- 1 (40 pts.) (a) Find the projection matrix P_C onto the column space of A (after looking closely at the matrix!)

$$A = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

- (b) Find the 3 by 3 projection matrix P_R onto the row space of A . What is the closest vector in the row space to the vector $\mathbf{b} = (1, 0, 0)$? -3
- (c) Multiply $P_C A$ and then $P_C A P_R$. Your answers should be a little surprising—can you explain?
- (d) Find a basis for the subspace of all vectors orthogonal to the row space of A .

a. The cols of A aren't independent. $B = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ is a basis.

$$B \text{ is just a vector, so } P_C = \frac{B B^T}{B^T B}$$

$$= \frac{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \end{bmatrix}}{\begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}} = \frac{\begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}}{9+1} = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} = P_C$$

b. P_R projects onto the column space of A^T

$A^T = \begin{bmatrix} 3 & 1 \\ 3 & 1 \\ 6 & 2 \end{bmatrix}$. These columns are also not independent, so use the basis $C = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
Again, C is just a vector, so use:

$$P_R = \frac{C C^T}{C^T C} = \frac{\begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \end{bmatrix}}{\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}} = \frac{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}{1+4}$$

$$P_R = \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

(2) $\frac{1}{2} \ln 2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(3) $\frac{1}{2} \ln 2$

(4) $\frac{1}{2} \ln 2$

(5) $\frac{1}{2} \ln 2$

(6) $\frac{1}{2} \ln 2$

(7)

$$c. \quad P_C A = \frac{1}{10} \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$

$$= \frac{1}{10} \begin{bmatrix} 30 & 30 & 60 \\ 10 & 10 & 20 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = A \quad \checkmark$$

$$P_C A P_R = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} \cdot \frac{1}{6} \begin{bmatrix} 1 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 18 & 18 & 36 \\ 6 & 6 & 18 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 6 \\ 1 & 1 & 2 \end{bmatrix} = A \quad \checkmark$$

We find that $P_C A = A P_R = P_C A P_R = A$.

This is not surprising; P_C projects vectors onto the colspace of A . Every column of A is in its colspace by definition, so $P_C A = A$.

$A P_R = (P_R^T A^T)^T = (P_R A^T)^T$, because a projection matrix is its own transpose.

$P_R A^T$ projects A^T 's columns onto $C(A^T)$.

So $(P_R A^T)^T = (A^T)^T = A$. It follows that

$P_C A P_R$ also equals A .

d. The nullspace is orthogonal to the row space. It is a 2-d space in \mathbb{R}^3 defined by the

basis vectors $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ \checkmark

$$P_D = \frac{1}{3} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 9 & 4 & 3 \\ 4 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}$$

- 2 (30 pts.) (a) Choose c and the last column of Q so that you have an orthogonal matrix:

$$Q = c \begin{bmatrix} 1 & -1 & -1 & x \\ -1 & 1 & -1 & x \\ -1 & -1 & -1 & x \\ -1 & -1 & 1 & x \end{bmatrix}$$

- (b) Project $b = (1, 1, 1, 1)$ onto the first column of Q . Then project b onto the plane spanned by the first two columns.
- (c) Suppose the last column of the 4 by 4 matrix (where the x 's are) was changed to $(1, 1, 1, 1)$. Call this new matrix A . If Gram-Schmidt is applied to the 4 columns of A , what would be the 4 outputs q_1, q_2, q_3, q_4 ? (Don't do a lot of calculations... please.)

a. To ensure all columns have length 1, $c = 1/2$. The last column must be $\begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$ for it to be orthogonal to the other three columns; note that this new column's dot product w/ the other three is zero.

b.
$$\vec{p}_1 = \frac{q_1 q_1^T b}{q_1^T q_1} = c \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{-2 \cdot c}{4c^2} = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$
 (onto the first column)

Since the columns of Q are orthogonal, we need only project b onto the second column of Q and add the resulting vectors.

$$q_4 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

This is already unit length

$$q_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ -1/2 \end{bmatrix} - \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$P_3 = \frac{q_3^T q_3 b}{q_3^T q_3} = \frac{1}{-1} \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

P_1, P_2, P_3 are projections onto q_1, q_2, q_3 . P_1 and P_2 are as above.

$$q_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} - P_1 - P_2 - P_3$$

where

To find q_4 , use Gram-Schmidt!

$$q_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}, \quad q_2 = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad q_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

The first three columns are already orthogonal! So Gram-Schmidt doesn't change them.

$$P_{12} = P_1 + P_2 = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \\ -1/2 & -1/2 \\ -1/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$P_2 = \frac{q_2^T q_2 b}{q_2^T q_2} = \frac{1}{2} \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1/4 \\ -1/4 \\ 1/4 \end{bmatrix}$$

- 3 (30 pts.) (a) If you multiply all $n!$ permutations together into a single P , is the product odd or even? (Answer might depend on n .)
- (b) If you know that $\det A = 6$, what is the determinant of B ?

$$\det A = \begin{vmatrix} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{vmatrix} = 6 \qquad \det B = \begin{vmatrix} \text{row 3} + \text{row 2} + \text{row 1} \\ \text{row 2} + \text{row 1} \\ \text{row 1} \end{vmatrix} = ?$$

- (c) Prove $\det A = 0$ for the 5 by 5 all-ones matrix (all $a_{ij} = 1$) in **two ways**:

- (1) Using Properties 1-10 of determinants
- (2) Using the "big formula" = sum of 120 terms.

a. The permutation matrices have determinant ± 1 , because they consist of I with row exchanges. Thus the determinant of their product ~~can~~ only be odd. It cannot be even. (4)

b. $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot A$

$$\det B = \det \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \cdot \det A$$

$$= (1 \cdot 0 - 1 \cdot 0 + 1 \cdot -1) \cdot \det A$$

$$= -1 \cdot \det A = \boxed{-6}$$

C. 1) Call the determinant d . d is unchanged when one row is subtracted from another. Subtract the last row from the others, giving one row of ones and four rows of zeros. This matrix has determinant d . The first row of the matrix has only zeros. Since the determinant is linear in the first row, d must be zero.

2) The determinant has 120

terms of $\pm 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = \pm 1$

Half of these terms require an even number of row exchanges, so they are positive; the other half are negative because they require an odd number.

The sum must be zero