18.06 Professor Strang/Ingerman Quiz 3 December 6, 2002

Your name is: DAN PORTS

Please circle your recitation:

1)	M2	2-131	P.-O. Persson	2-088	2-1194	persson
2)	M2	2-132	I. Pavlovsky	2-487	3-4083	igorvp
3)	M3	2-131	I. Pavlovsky	2-487	3-4083	igorvp
4)	T10	2-132	W. Luo	2-492	3-4093	luowei
5)	T10	2-131	C. Boulet	2-333	3-7826	cilanne
6)	T11	2-131	C. Boulet	2-333	3-7826	cilanne
7)	T11	2-132	X. Wang	2-244	8-8164	xwang
8)	T12	2-132	P. Clifford	2-489	3-4086	peter
9)	T1	2-132	X. Wang	2-244	8-8164	xwang
10)	T1	2-131	P. Clifford	2-489	3-4086	peter
11)	T2	2-132	X. Wang	2-244	8-8164	xwang

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36

1 (36 pts.) 12 (a) What are the eigenvalues of the 5 by 5 matrix $A = \boldsymbol{ones}(5)$ with all entries $a_{ij} = 1$? Please look at A, not at $\det(A - \lambda I)$.

12 (b) Solve this differential equation to find $\boldsymbol{u}(t)$:

$$\frac{d\boldsymbol{u}}{dt} = A\boldsymbol{u} \qquad \text{starting from } \boldsymbol{u}(0) = (0, 1, 1, 1, 2)\,.$$

First split $\boldsymbol{u}(0)$ into two eigenvectors of A.

12 (c) Using part (a), what are the *eigenvalues* and *trace* and *determinant* of the matrix $B =$ same as A except zeros on the diagonal.

a. A is singular, so one eigenvalue is zero. A has rank 1, so the nullspace = the $\lambda = 0$ eigenspace has dimension $5-1=4$, so 0 is repeated 4 times.

We also have $\begin{bmatrix} 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \\ 1&1&1&1&1 \end{bmatrix}\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix} = \begin{bmatrix}5\\5\\5\\5\\5\end{bmatrix} = 5\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$, so $\lambda = 5$.

$\boxed{\lambda = 0, 0, 0, 0, 5}$

b. $u(0) = \begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} + \begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix}$, the $\lambda = 5$ evector and a $\lambda = 0$ e-vec.

$u(t) = S e^{\Lambda t} S^{-1} u(0)$

$u(t) = e^{5t}\begin{pmatrix}1\\1\\1\\1\\1\end{pmatrix} + e^{0t}\begin{pmatrix}-1\\0\\0\\0\\1\end{pmatrix} = \boxed{\begin{pmatrix} e^{5t} - 1 \\ e^{5t} \\ e^{5t} \\ e^{5t} \\ e^{5t} + 1 \end{pmatrix}}$

c. $\boxed{\operatorname{tr} B = \text{sum of diagonal} = 0}$

$B = \begin{bmatrix} 0&1&1&1&1 \\ 1&0&1&1&1 \\ 1&1&0&1&1 \\ 1&1&1&0&1 \\ 1&1&1&1&0 \end{bmatrix}$ $\quad B - {-1}I = A \Rightarrow -1$ is an eigenval because $\det A = 0$.

This eigenvalue is repeated 4 times because A has $\lambda = 0$ repeated 4 times. For the trace to be zero the fifth e-val must be 4.

(over)

c.

$$\boxed{\begin{aligned} &\text{So } \lambda_B = -1, -1, -1, -1, 4 \\ &\operatorname{tr} B = 0 \\ &\det B = -1 \cdot -1 \cdot -1 \cdot -1 \cdot 4 = 4 \end{aligned}}$$

18

2 (20 pts.) (a) If A is similar to B show that e^A is similar to e^B. *First define "similar" and* e^A!!

(b) If A has 3 eigenvalues $\lambda = 0, 2, 4$, *find the eigenvalues of* e^A.

Using part (a) explain this connection with determinants:

$$\text{determinant of } e^A = e^{\text{trace of } A}$$

a. A and B similar means A and B have the same eigenvalues; $B = M^{-1}AM$

$$e^A = I + A + \frac{A^2}{2} + \frac{A^3}{3!} \cdots$$

If A and B have the same eigenvalues, then all powers of A and B have the same eigenvalues.

$$e^B = I + M^{-1}AM + \frac{M^{-1}AMM^{-1}AM}{2} + \cdots$$

$$= I + M^{-1}AM + \frac{M^{-1}A^2M}{2} + \frac{M^{-1}A^3M}{3!}$$

$$= M^{-1}(I + A + A^2/2 + A^3/3! \cdots)M$$

$$= M^{-1}e^A M$$

$\Rightarrow e^B$ similar to e^A.

b. A is similar to its Jordan form $J = \begin{bmatrix} 0 & & \\ & 2 & \\ & & 4 \end{bmatrix}$. This is diagonal, so $e^J = \begin{bmatrix} e^0 & & \\ & e^2 & \\ & & e^4 \end{bmatrix} = \begin{bmatrix} 1 & & \\ & e^2 & \\ & & e^4 \end{bmatrix}$. The e-vals are on the diagonal: $\{1, e^2, e^4\}$

Provided A is diagonalizable (it is in this case),

$A = S\Lambda S^{-1} \Rightarrow A$ similar to $\Lambda \Rightarrow e^\Lambda$ similar to e^A

$e^{\text{tr} A} = e^{\lambda_1 + \lambda_2 + \lambda_3}$

$\det e^A =$ product of e^A's evals

and in other cases? (-2)

$=$ product of e^A's evals

$e^A = \begin{bmatrix} e^{\lambda_1} & & \\ & e^{\lambda_2} & \\ & & e^{\lambda_3} \end{bmatrix}$, so its determinant, the product of its e-val's, is $e^{\lambda_1} e^{\lambda_2} e^{\lambda_3}$

$= e^{(\lambda_1 + \lambda_2 + \lambda_3)}$, which was the same as $e^{(\mathrm{tr}\, A)}$, so $e^{\mathrm{tr}\, A} = \det(e^A)$.

20

3 (22 pts.) Suppose the SVD $A = U\Sigma V^{\mathrm{T}}$ is

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

9 (a) For which angles θ and α (0 to $\frac{\pi}{2}$) is A a positive definite symmetric matrix? No computing needed.

11 (b) What are the eigenvalues and eigenvectors of $A^{\mathrm{T}}A$? No computing!

a. A pos. def, symmetric matrix has

$A = Q\Lambda Q^T$, with positive eigenvals.

The singular values of A (from the matrix) are 9 and 4. Those are the square roots of the e-vals of A^TA, so if A is symmetric they must be the e-vals of A. If A is to be symmetric then $A = U\Sigma V^T = Q\Lambda Q^T$

$\Rightarrow Q = U$ and $V^T = Q^T$ So V^T must be U^T

$\Rightarrow V = U$. $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$\Rightarrow \boxed{\theta = \alpha}$ ✓ for symmetry. This

also gives positive definiteness since the eigenvals will be $\cancel{\pm}9$ and $\cancel{\pm}4$, both positive.

Any value of $0 \le \theta = \alpha \le \pi/2$ will work.

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(over)

3B. $A = U\Sigma V^T$

$\Rightarrow A^TA = V\Sigma^T U^T U \Sigma V^T = V\Sigma^2 V^T$ since Σ is diagonal

This is a $Q \Lambda Q^T$ decomposition for A^TA, so A^TA has eigenvalues that are the singular values squared $\boxed{(9^2 \text{ and } 4^2)}$ and eigenvectors that are the columns of V: $\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}$ and $\begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$

22

4 (22 pts.) Multinational companies in the US, Asia, and Europe have assets of \$ 12 trillion. At the start, \$ 6 trillion are in the US, \$ 6 trillion in Europe. Each year half the US money stays home, $\frac{1}{4}$ each goes to Asia and Europe. For Asia and Europe, half stays home and half is sent to the US.

$$\begin{bmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k+1} = \begin{bmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{bmatrix} \begin{bmatrix} \text{US} \\ \text{Asia} \\ \text{Europe} \end{bmatrix}_{\text{year } k}$$

11 (a) The eigenvalues and eigenvectors of this *singular* matrix A are

$\boxed{\lambda_1 = \lambda_{max} = 1}$ because A is a Markov matrix.

$A - \lambda I = \begin{bmatrix} -.5 & .5 & .5 \\ .25 & -.5 & 0 \\ .25 & 0 & -.5 \end{bmatrix} \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{x_1} = 0$

(over)

11 (b) The limiting distribution of the \$ 12 trillion as the world ends is

US = \$ 6T

Asia = \$ 3T

Europe = \$ 3T

$A^\infty \begin{bmatrix} \$6T \\ \$6T \\ 0 \end{bmatrix}$ decompose $\begin{bmatrix} \$6T \\ \$6T \\ 0 \end{bmatrix}$ into a combo of eigenvectors: $\$3T \underbrace{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}_{x_1} + \$3T \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{x_2}$

The x_1 component goes to zero as $A \to A^\infty$ because its e-val is less than 1. The x_1 component is preserved unchanged because its e-val is 1.

So $A^\infty \begin{bmatrix} \$6T \\ \$6T \\ 0 \end{bmatrix} = \$3T \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \$6T \\ \$3T \\ \$3T \end{bmatrix}$

4a. $\boxed{\lambda_2 = 0}$ because the matrix is singular

$$\begin{bmatrix} .5 & .5 & .5 \\ .25 & .5 & 0 \\ .25 & 0 & .5 \end{bmatrix} \boxed{\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}_{x_2}} = 0$$

$\boxed{\lambda_3 = .5}$ because $\operatorname{tr} A = \lambda_1 + \lambda_2 + \lambda_3 = 1.5$.

$$A - .5I = \begin{bmatrix} 0 & .5 & .5 \\ .25 & 0 & 0 \\ .25 & 0 & 0 \end{bmatrix} \boxed{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}_{\lambda_3}}$$