18.100B Midterm Exam
March 21, 2003

Name: Dan Ports 87

No books, calculators, etc. allowed. Show all work in the area provided. Indicate when appealing to established results.

Problem 1. For (x_1, y_1) and (x_2, y_2) in $\mathbb{R}^2$, define $d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|$. Show that d is a metric (sometimes called the city-block metric), and describe the neighborhood of radius 1 about the point $(2, 2)$.

We show that d satisfies the 3 metric axioms

let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ ...

1) $d(P_1, P_2) \geq 0$ $\Rightarrow$? $|x_1 - x_2| + |y_1 - y_2| > 0$ if $x_1 \neq x_2$ or $y_1 \neq y_2$.

This is clearly true because both terms are absolute values, which are always non-negative.

$d(P_1, P_2) = 0$ only when $P_1 = P_2$. If $d(P_1, P_2) = 0$,

$|x_1 - x_2| + |y_1 - y_2| = 0 \Rightarrow |x_1 - x_2| = 0$ and $|y_1 - y_2| = 0$

by non-negativity of absolute values $\Rightarrow x_1 = x_2$ and $y_1 = y_2 \Rightarrow P_1 = P_2$

2) $d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2| = |x_2 - x_1| + |y_2 - y_1| = d(P_2, P_1)$

3) $d(P_1, P_2) \leq d(P_1, P_3) + d(P_3, P_1)$

$\Leftrightarrow$ $|x_1 - x_2| + |y_1 - y_2| \leq |x_1 - x_3| + |y_1 - y_3| + |x_3 - x_2| + |y_3 - y_2|$.

We know $|x_1 - x_2| \leq |x_1 - x_3| + |x_3 - x_2|$ by the triangle inequality in the standard Euclidean metric in $\mathbb{R}^1$ and similarly w/ y replacing x. Adding these two inequalities gives the desired triangle inequality in $(\mathbb{R}^2, d)$.

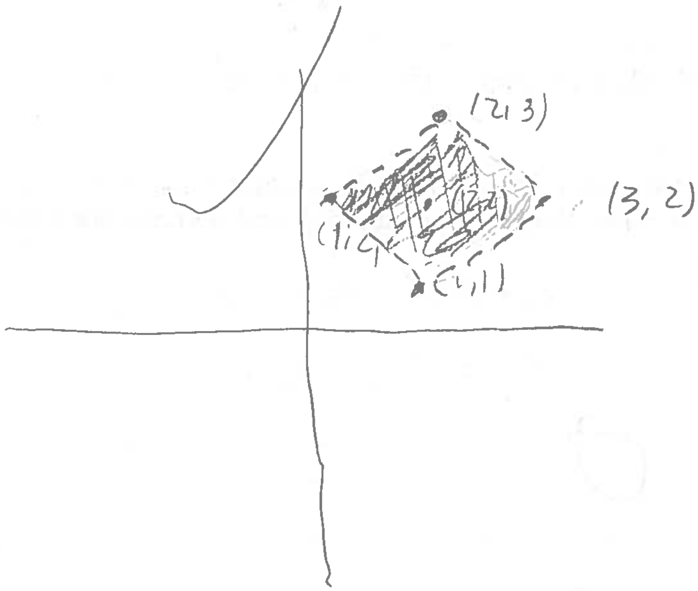
d satisfies all 3 metric axioms, so it is a metric.

over $\Rightarrow$

The neighborhood w/ r=1 about (2,2)

is $|x_1 - 2| + |y_1 - 2| < 1$,

This is the interior of a diamond!



Problem 2. Suppose that A and B are nonempty compact subsets of $\mathbb{R}^n$ with $A \cap B = \emptyset$. Let $E = \{d(p,q) \mid p \in A,\ q \in B\}$. Show that $inf(E) > 0$.

Suppose $\inf(E)$ were not > 0. It cannot be negative, since $d(p,q) \geq 0$, so $\inf(E)$ must be zero. This means that 0 is the glb of $d(p,q)$. If there existed a $p \in A$, $q \in B$ such that $d(p,q) = 0$, then $p = q$ and p would be in both A and B, contradicting their disjointness.

So we know that $\forall \varepsilon > 0$ we can find a $p \in A$, $q \in B$ such that $d(p,q) = \varepsilon$. (If this were not true, $\inf(E)$ could not be zero as we are assuming).

what if A and B have no limit pts.?

Thus either A contains a limit point of B or B contains a limit point of A. (If this were not true, then there would be an $r > 0$ such that $\forall p \in A$, $N_r(p) \cap B = \emptyset \Rightarrow \forall p \in A, \forall q \in B, d(p,q) > r$, and likewise with A and B reversed. But this would contradict our above finding that $\forall \varepsilon > 0$, $\exists p \in A, q \in B$ such that $d(p,q) < \varepsilon$.)

But A and B are compact, so they are closed, so they contain their limit points. This means that there is a point p in both A and B, which contradicts their disjointness.

18/20

Thus by contradiction, we have shown $\inf(E) > 0$

Problem 3.

(a) Answer True or False:

F i. $\mathbb{C}$ is an ordered field.

-2 ~~T~~ ii. If A and B are bounded sets of real numbers, and $C = \{ab \mid a \in A,\ b \in B\}$, then $sup(C) = sup(A)sup(B)$.

T iii. A compact metric space is complete.

T iv. Every bounded sequence in $\mathbb{R}^n$ contains a convergent subsequence.

F v. If E is an uncountable subset of $\mathbb{R}$ then E must contain an interval.

-2 ~~F~~ vi. If E is a compact subset of a metric space X then E is closed and bounded.

(b) Complete the sentence: A subset E of a metric space (X, d) is compact if and only if ...

every open cover of E contains a finite subcover

(c) Complete the definition: If X is a metric space and E is a subset of X, then E is dense in X if ...

every point in X is either a point of E or a limit point of E.

(d) Complete the definition: An ordered set S has the least upper bound property provided ...

If any subset $E \subset S$ is bounded above, then it has a least upper bound in S.

-1

15/20

Problem 4. Let E be a connected set in a metric space (X, d). Prove that the closure of E is connected.

Suppose $\bar{E}$ were not connected. Then it can be written as $A \cup B$ for separated A, B. Note that since $\bar{E} = E \cup E'$, $E \subseteq \bar{E}$, So every point $p \in E$ is in $\bar{E}$, and thus either $p \in A$ or $p \in B$. $E \subseteq (A \cup B)$

$\Rightarrow E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B)$. If A and B are separated, then $(E \cap A)$ and $(E \cap B)$ are separated (we know that subsets of separated spaces are separated),

We have thus shown that E is the union of two separated sets. This contradicts the connectedness of E, so $\bar{E}$ must be connected

$\frac{20}{20}$

Problem 5. Let $\{a_n\}$ be a bounded sequence of real numbers and suppose that $\sum_{n=1}^{\infty} a_n$ diverges. Show that the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n z^n$ is equal to 1.

$\alpha = \limsup_{n \to \infty} \sqrt[n]{a_n}$. We know α cannot be less than 1, because if so $\sum a_n$ would be convergent by the root test.

Since $\{a_n\}$ is a bounded sequence of real numbers,

False!

~~it must converge to a limit point in $\mathbb{R}$.~~ This requires that $\lim_{n \to \infty} a_n = c$ for some $c \in \mathbb{R}$.

If α were > 1, then there exists some subsequence a_{b_n} such that $\lim_{n \to \infty} \sqrt[n]{a_{b_n}} > 1$. Thus $\forall n > N$,

$\sqrt[n]{a_{b_n}} > 1 \Rightarrow \lim_{n \to \infty} a_{b_n} = (1+\varepsilon)^n$ for some $\varepsilon > 0$.

This limit is infinite (divergent geom. seq.), so some subsequence of a_n diverges, so a_n cannot converge, which contradicts its boundedness. Thus $\alpha = 1$.

?

The radius of convergence $R = \frac{1}{\alpha} = \frac{1}{1} = 1$

If $\lim_{n \to \infty} a_n = c$ then $\alpha = 1$?

$\frac{14}{20}$