2. A is a subset of $\mathbb{R}$, so it has the glb property. Thus A has an infimum, since we are given that it is bounded below. $-A$ must be bounded above by $-\inf A$.
Proof by contradiction. Suppose there were a $-x \in -A$ such that $-x > -\inf A$. Then x exists in A, by definition of $-A$. It follows from the ordered field axioms that $-x > -\inf A$ implies $x < \inf A$. But since $x \in A$, it cannot be less than $\inf A$. This is a contradiction, so there is no such x and $-A$ is bounded above by $-\inf A$.
$-\inf A$ is also the least upper bound on $-A$. Again, by contradiction: suppose there was an element $-x \in \mathbb{R}$, $-x < -\inf A$, such that every element of $-A$ is less than $-x$. The ordered field axioms imply that every element of A is greater than x (reversing the direction of the inequality and changing signs). This means x is a lower bound on A. Since $-x < -\inf A$, $x > \inf A$. Thus, x is a lower bound on A that is larger than the infimum, which is a contradiction. No such x can exist, so $-\inf A$ is the least upper bound on $-A$.
We have $-\inf A = \sup(-A)$; it trivially follows that $\inf A = -\sup(-A)$

✓

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3. a. We are given that $m/n = p/q$. This means $mq = pn$; mq and pn are integers.

$$[((b^m)^{1/n})^{nq} = b^{mq}$$ by the definition of $x^{1/n}$ and properties of integer exponents

$$= b^{pn}$$ from above

$$= [((b^p)^{1/q}]^{qn}$$

Taking the nqth root, which we know is unique, we have $(b^m)^{1/n} = (b^p)^{1/q}$.

b. Let $r = p/q$ and $s = c/d$

$$b^{r+s} = b^{p/q + c/d} = b^{(pd+qc)/qd}$$

$(pd+qc)$ and qd are both integers, since they are sums and products of integers, so we can use the property of integer exponents $b^{r+s} = b^r b^s$; we can rewrite the above expression as

$$= (b^{pd} b^{qc})^{1/qd}$$

$$= (b^{pd})^{1/qd} \cdot (b^{qc})^{1/qd}$$ from the uniqueness property

$$= b^{p/q} \cdot b^{c/d}$$

$$= b^r b^s$$

c. For $x, y \in \mathbb{Q}$ and $b > 1$, $x < y \Rightarrow b^x < b^y$

Proof: x and y are rational, so they can be written as $x = \frac{p}{z}$ and $y = q/z$, with $p, q \in \mathbb{Z}$. p will be less than q. Since b is greater than 1, and $q - p$ is a positive integer, $b^{q-p} > 1$, which implies $b^q > b^p$. This inequality is preserved when taking the zth root:

$(b^q)^{1/z} > (b^p)^{1/z} \Leftrightarrow b^y > b^x \Leftrightarrow b^x < b^y$

Consider any value $t \leq r$. From above, $b^t < b^r$. Since $B(x)$ is made up of those b^t values, we have shown that b^r is an upper bound on $B(x)$. Furthermore, it is the least upper bound, because any $n < b^r$, $n \geq 0$ can be written as $n = b^t$ with $t < r$, and this is not an upper bound because the density property allows us to choose a m such that $t < m < r$. b^m will be in $B(r)$ and $b^m > b^t = n$, so any $n < r$ cannot be an upper bound. (Any $n < 0$ is clearly not an upper bound because $B(x)$ contains positive values).
Thus $b^r = \sup B(r)$

d. $b^{x+y} = \sup B(x+y)$

Any element of $B(x+y)$ has form b^n, $n \in \mathbb{Q}$, $n < x+y$. There exists a $p, q \in \mathbb{Q}$ such that $p+q = n$ and $p < x$ and $q < y$. Proof:
Since $n < x+y$, $n - x < y$. By density of $\mathbb{Q}$, we can choose q between $n-x$ and y so $n - x \leq q < y$.
Then let $p = n - q$. $q \geq n - x$, so $p = n - q < n - n + x = x$.
Since $p \leq x$, b^p is a member of $B(x)$ and similarly b^q is a member of $B(y)$.
Since p and q are rationals, $b^{p+q} = b^p b^q$ from part b. $p + q = n$, and n is an element of $B(x+y)$. So we have shown that $B(x+y)$ is a subset of the set of products of an element from $B(x)$ and one from $B(y)$.

Next we show that any element in this set of products is also in $B(x+y)$. Consider any $p, q \in \mathbb{Q}$, $p < x$ and $q < y$. $b^p < b^x$, so b^p

3d cont'd.

so b^{p+q} is in $B(x+y)$. It is thus not greater than $\sup B(x+y) = b^{x+y}$. b^p and b^q are arbitrary elements of $B(x)$ and $B(y)$, so we have shown that the product of any element of $B(x)$ and any element of $B(y)$ is in $B(x+y)$. We previously showed that any element of $B(x+y)$ is a product of an element of $B(x)$ and one of $B(y)$. So $B(x+y)$ and the set of products are the same, and

$\Rightarrow \sup B(x+y) = \sup B(x) \cdot \sup B(y)$

$\Rightarrow b^{x+y} = b^x b^y$ ✓

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4. The lexicographic ordering turns the set of all complex numbers into an ordered set.

Proof:

Consider an arbitrary $z = a+bi$, $w = c+di$.
The real numbers are an ordered set, so either $a<c$, $a=c$, or $a>c$.
If $a<c$, $z<w$ by the lexicographic ordering.
If $a>c$, $z>w$
Otherwise $a=c$.
In this case, we compare b and d. Since they are real, and the real numbers are an ordered set, either $b<d$, $b=d$, or $b>d$.
If $b<c$, the definition of the ordering gives $z<w$
$b>c$, it gives $z>w$
Otherwise $b=d$ and $z=w$.
So for any z and any w, one and only one of $z<w$, $z=w$, or $z>w$ is true.
The lexicographic ordering also satisfies the transitive property. Consider $z=a+bi$, $w=c+di$, and $x=m+ni$, and suppose that $z<w$ and $w<x$. We show that this leads to $z<x$.
Since $z<w$, either $a<c$ or $a=c$ and $b<d$, and similarly either $c<m$ or $c=m$ and $d<n$.
By cases:
1) $a<c$, $c<m$. Because the real numbers are an ordered set, $a<m$ and $z<x$ by def.
2) $a=c$, $b<d$, $c<m$. Thus $a<m$ and by definition $z<x$
3) $a<c$, $c=m$, $d<n$ $\Rightarrow$ $a<m$ so $z<x$
4) $a=c$, $b<d$, $c=m$, $d<n$ $\Rightarrow$ $a=c=m$, $b<d<n$ so $z<x$ under the lexicographic ordering.
Thus $z<w$, $w<x$ $\rightarrow$ $z<x$. The lexicographic

It does not have the LUB property.
Consider $A = \{a + bi \mid a < 0\}$.
A is clearly bounded above by $0 + 0i$, but there is no LUB.

Proof: Suppose there was a LUB $a + bi$. a must equal zero, since the real parts of the elements in A are < 0. But for any value of b, we can choose $0 + (b-1)i$ instead giving a number which is also a lexicographic upper bound on A, but lexicographically less than $0 + bi$, which is a contradiction since we chose that to be the least upper bound. Thus there is no LUB.

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5. a) $\mathbb{Q}(\sqrt{2})$ is a subset of $\mathbb{R}$ because every $x \in \mathbb{Q}(\sqrt{2})$ has form $a+b\sqrt{2}$, where $a,b \in \mathbb{Q}$. This means $a,b \in \mathbb{R}$, and we know $\sqrt{2} \in \mathbb{R}$, so $a+b\sqrt{2}$ is also in $\mathbb{R}$ because $\mathbb{R}$ is a field and is closed under addition and multiplication. Because $\mathbb{R}$ is a field, and $\mathbb{Q}(\sqrt{2})$'s set is a subset of $\mathbb{R}$'s set, it suffices to show that $x+y$, $-x$, xy, and $1/x$ are well-defined in $\mathbb{Q}(\sqrt{2})$ because all the field axioms that held in $\mathbb{R}$ must also hold in $\mathbb{Q}(\sqrt{2})$ as long as these quantities exist in $\mathbb{Q}(\sqrt{2})$.

So consider $x = a+b\sqrt{2}$, $y = c+d\sqrt{2}$,

$x+y = a+b\sqrt{2}+c+d\sqrt{2} = (a+c)+(b+d)\sqrt{2}$, which is in $\mathbb{Q}(\sqrt{2})$.

$-x = -(a+b\sqrt{2}) = -a-b\sqrt{2}$, which is in $\mathbb{Q}(\sqrt{2})$

$xy = (a+b\sqrt{2})(c+d\sqrt{2}) = ac + ad\sqrt{2} + bc\sqrt{2} + b\sqrt{2}\,d\sqrt{2}$

$= (ac+2bd) + (ad+bc)\sqrt{2}$, which is in $\mathbb{Q}(\sqrt{2})$

$1/x = \frac{1}{(a+b\sqrt{2})} = \frac{a-b\sqrt{2}}{(a+b\sqrt{2})(a-b\sqrt{2})} = \frac{a-b\sqrt{2}}{a^2-2b^2}$

✓ $= \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}$, which is in $\mathbb{Q}(\sqrt{2})$

b). Suppose $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$.

Then $\sqrt{3} = a+b\sqrt{2}$

$\Rightarrow 3 = (a+b\sqrt{2})^2 = a^2 + \sqrt{2}ab + b^2$

$\Rightarrow 3 - a^2 - b^2 = \sqrt{2}\,ab$

We know that a and b are raitonal numbers. If they are both non-zero, then we can divide by ab and have

$\sqrt{2} = \frac{3-a^2-b^2}{ab}$ The numerator is rational, and so is the denominator, so $\sqrt{2}$ must be rational, which is a contradiction. If $a \neq 0$ and $b \neq 0$, Which is the same contradiction. Finally, if $a = b = 0$, then we have $3 = 0$, an obvious contradiction. So our assumption that $\sqrt{3} \in \mathbb{Q}(\sqrt{2})$ is proven false.

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6. Consider the sets M_n consisting of the subsets of $\{x \in \mathbb{Z}, |x| \leq n\}$. For any value of n, the set M_n is finite. Specifically, it contains 2^{2n+1} elements, because each integer with absolute value $\leq n$ is either in the subset or not (from elementary combinatorics). The set M is the union $\bigcup_{\alpha \in J} M_\alpha$. This is the union of a countable set of finite sets, so it is countable. (Theorem 2.12) ✓

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7. We know that $\mathbb{Q}$ is countable. The set of polynomials with degree n-1 has a bijection to the set $\mathbb{Q}^{n+1}$ (represent the coefficients as a n+1-tupled. This set is countable (Rudin, thm 2.13). Now consider the set A_n of solutions to nth-degree polynomial equations. Since each equation of degree n has at most n solutions, there are a finite number of solutions to each degree-n equation. Thus A_n is a countable set, because it is the union of a countable set of finite sets. ✓ The set of algebraic numbers is the set of solutions to equations of any degree;

$$A = \bigcup_{n=0}^{\infty} A_n$$

This is a union of a countable set of countable sets, so it is countable.

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