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18.100B PS2

1. a. d_1 is not a metric: it does not satisfy the triangle inequality:

$d_1(2,0) = |2-0|^2 = 2^2 = 4$

$\not\leq d_1(2,1) + d_1(1,0) = |2-1|^2 + |1-0|^2 = 2+1 = 3$

b. d_2 is not a metric. Observe that

$d(p,-p) = |p^2 + (-p)^2| = |p^2 - p^2| = 0$.

Since $p \neq -p$ (for $p \neq 0$), d_2 cannot be a metric on $\mathbb{R}$.

c. d_3 is a metric.

1) $d(x,y) \geq 0$: $\frac{|x-y|}{1+|x-y|}$ is always positive since $|x-y|$ is positive, $1+|x-y|$ is positive, and so the numerator and denom. are both positive. It is only zero for $|x-y| = 0 \Rightarrow x = y$.

2) $d(x,y) = \frac{|x-y|}{1+|x-y|} = \frac{|-(x-y)|}{1+|-(x-y)|} = \frac{|y-x|}{1+|y-x|} = d(y,x)$

3) $d(x,z) \leq d(x,y) + d(y,z)$

$\Rightarrow \frac{|x-z|}{1+|x-z|} \leq \frac{|x-y|}{1+|x-y|} + \frac{|y-z|}{1+|y-z|}$

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$$\Rightarrow |x-z|(1+|x-y|)(1+|y-z|) \le |x-y|(1+|x-z|)(1+|y-z|) + |y-z|(1+|x-z|)(1+|x-y|)$$

$$\Rightarrow |x-z| + \cancel{|x-z||x-y|} + \cancel{|x-z||y-z|} + \cancel{|x-z||x-y||y-z|}$$
$$\le |x-y| + \cancel{|x-y||x-z|} + |x-y||y-z| + \cancel{|x-y||y-z||x-z|}$$
$$+ |y-z| + \cancel{|y-z||x-z|} + |y-z||x-y| + |y-z||x-z||x-y|$$

$$\Rightarrow |x-z| \le |x-y| + |y-z| + 2|x-y||y-z| + |x-z||x-y||y-z|$$

We know the inequality holds because $|x-z| \le |x-y| + |y-z|$ is the triangle inequality for the standard Euclidean metric in $\mathbb{R}^1$, and the other terms are positive. So the triangle inequality holds here.

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2. a) d_∞ satisfies the 3 criteria for a metric

1) $d_\infty \geq 0$ because each element of the set $A = \{|p_1 - q_1|, |p_2 - q_2|, \ldots, |p_n - q_n|\}$ is an absolute value and is greater than zero. It is only equal to zero when $p = q$; if $p \neq q$, then the coordinates differ in at least one position, so at least one element is non-zero and the supremum cannot be zero.

2) $d(p,q) = d(q,p)$. Each element is $|p_i - q_i|$, which is equivalent to $|q_i - p_i|$. The supremum doesn't change when p and q are interchanged.

3) $d(x,z) \leq d(x,y) + d(y,z)$ Proof: Designate the indices of the maximal elements of $|x_i - z_i|$, $|x_i - y_i|$, and $|y_i - z_i|$ as a, b, and c respectively. Then the triangle inequality becomes $|x_a - z_a| \leq |x_b - y_b| + |y_c - z_c|$. We know that $|x_a - z_a| \leq |x_a - y_a| + |y_a - z_a|$ (*) as a consequence of the triangle inequality for the standard Euclidean metric in $\mathbb{R}^1$. We also know that $|x_b - y_b|$ is at least as large as $|x_a - y_a|$, because if it were not, then it could not be the supremum of the set because $|x_a - y_a|$ is larger. By the same argument, $|y_c - z_c|$. Combining this with (*) above, we have

$$|x_a - z_a| \leq |x_b - y_b| + |y_c - z_c| \Rightarrow d(x,z) \leq d(x,y) + d(y,z)$$

b. We show that E is open in $(\mathbb{R}^n, d)$ iff E is open in $(\mathbb{R}^n, d_\infty)$.

Pf: ($\Rightarrow$) Suppose $E \subseteq \mathbb{R}^n$ is open under metric d. Then because E is open, by definition, $\forall p \in E \ \exists r > 0$ so that the nbhd $N_{r,d}(p) \subseteq E$. Specifically, the nbhd $N_{r,d}(p)$ consists of all points $q \in \mathbb{R}^n$ such that $d(p,q) = \|p - q\| < r$. We can then choose $r' \leq r/\sqrt{n}$ and consider the nbhd $N_{r', d_\infty}(p)$ consisting of all points q such that

$d_\infty(p,q) = \sup\{p_1 - q_1, p_2 - q_2, \ldots p_n - q_n\} \leq r'$.

Any point $q \in N_{r', d_\infty}(p)$ is also in $N_{r,d}(p)$. Proof: $d_\infty(p,q) = \sup\{|p_1 - q_1|, |p_2 - q_2|, \ldots, |p_n - q_n|\} < r'$, so we know that every quantity $|p_i - q_i| < r'$ by def of supremum. So $d(p,q) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2}$ $\leq \sqrt{n r'^2} = r'\sqrt{n} = r/\sqrt{n}\,\sqrt{n} = r$. So q satisfies the requirements to be in the nbhd $N_{r,d}(p)$, and $N_{r,d}(p) \subseteq E$, so $q \in E$ and thus E is open under d_∞.

($\Leftarrow$) Next suppose $E \subseteq \mathbb{R}^n$ is open under metric d_∞. That is, $\forall p \in E$ $\exists r > 0$ so that $N_{r,d_\infty}(p) \subset E$. $N_{r,d_\infty}(p)$ is defined by $\forall q \in N_{r,d_\infty}(p)$, $d_\infty(p,q) < r$, or equivalently $\sup\{|p_1 - q_1|, |p_2 - q_2|, \ldots |p_n - q_n|\} < r$. Now consider $N_{r,d}(p)$, the set of points q such that $d(p,q) = \sqrt{\sum_{i=1}^{n}(p_i - q_i)^2} < r$. Every such point q also satisfies $d_\infty(p,q) < r$: all the terms in the summation are positive so the worst case is when they are all zero except one, and in this case it reduces to $\sqrt{(p_a - q_a)^2}$ (for some $a \leq n$). This shows that $|p_a - q_a| < r$, and clearly r is the supremum of the set of differences, so $d_\infty(p,q) < r$. Thus every point $q \in N_{r,d}(p)$ is also in $N_{r,d_\infty}(p)$, and since $N_{r,d_\infty}(p) \subset E$, $N_{r,d}(p) \subset E$ and thus E is open under d.

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3 a. $d(p,q)$ satisfies the 3 criteria:

1) $d(p,q) > 0$ if $p \neq q$ and $= 0$ if $p = q$, trivially from the definition

2) $d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases} = \begin{cases} 1 & \text{if } q \neq p \\ 0 & \text{if } q = p \end{cases} = d(q,p)$

3) $d(x,z) \leq d(x,y) + d(y,z)$

The left side of the inequality is at most 1, by definition. Unless $x=y=z$, the right side will either be 1 or 2; if $x=y=z$, then both sides will be zero. The inequality always holds.

So $d(p,q)$ is a metric ✓

b. Every subset A of (X, d) is both open and closed.

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Any A can have no limit points. Consider any point $p \in X$ and the neighborhood $N_{1/2}(p)$. The only element w/ distance $< 1/2$ to p is p itself. So this neighborhood contains no elements of A except possibly p, and so p cannot be a limit point of A. Any set has no limit points so it is vacuously closed.

Consider a subset $A \subseteq X$. Every $p \in A$ is an interior point. Consider $N_{1/2}(p)$; as before, we know this neighborhood consists only of p so $N_{1/2}(p) = \{p\} \subseteq A$. This is the definition of an interior point. Every point is an interior point, so every set A is open.

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4. Consider a point $p \in (E')^c$. This means p is not a limit point of E, so there must be at least one neighborhood $N_r(p)$ that does not contain any points of E other than p. No point $q \in N_r(p)$ can be a limit point of E, because if it were then there would be points of E in every neighborhood of q, and thus there would be members of E in $N_r(p)$, which is a contradiction. So every point p has a neighborhood $N_r(p) \subseteq (E')^c$, so $(E')^c$ is open and E' is closed.

Any limit point p of E will be a limit point of $\bar{E}$. It will have a neighborhood $N_r(p)$ that contains at least one point $q \in E$, and since $\bar{E} = E \cup E'$, $q \in \bar{E}$. Thus any $N_r(p)$ contains a point in $\bar{E}$, so p is a limit point of $\bar{E}$.

Now consider a limit point p of $\bar{E}$. Any nbhd $N_r(p)$ contains a point $q \in \bar{E}$. $\bar{E} = E \cup E'$, so either $q \in E$ or $q \in E'$. If $q \in E$, then $N_r(p)$ contains a point (q) in E. Otherwise $q \in E'$, which means any nbhd $N_R(q)$ contains points in E. So there must be some point in $N(p)$ that is in E. So any neighborhood of a limit point of $\bar{E}$ contains points in E: any limit point of $\bar{E}$ is a limit point of E. So E and $\bar{E}$ have the same limit points.

E and E' do not always have the same limit points. Consider $E \subseteq \mathbb{R}^1$, where $E = \{1/n, n \in \mathbb{N}\}$. The only limit point of E is 0, so $E' = \{0\}$. But E' obviously has no limit points, so $E' \neq E''$.

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5a. Consider an arbitrary element p in some set $\bar{A}_k$, for $0 < k \leq n$. Either $p \in A_k$, or $p \in A_k'$. If $p \in A_k$, then $p \in B$ since $B = \cup_{i=1}^{\infty} A_i$, and thus $p \in \bar{B}$. Otherwise, $p \in A_k'$. We show that p must then be a limit point of B. Since any neighborhood $N_r(p)$ is contained in A_k it must be contained in B. Thus $b \in B'$, so $b \in \bar{B}$.

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We next consider an element $p \in \bar{B}$, noting that $p \in B$ or $p \in B'$. If $p \in B$, it must exist in some A_k, so $p \in A_k \Rightarrow p \in \cup_{i=1}^{n} \bar{A}_i$. If $p \in B'$, then any nbhd $N_r(p)$ contains an element $q \in B$, and so q must exist in some A_i by the definition of B. Since there are infinite such neighborhoods and only finitely many sets A_i, there must exist some set A_k where every neighborhood of p contains an element in A_k (via the pigeonhole principle), so p is a limit point of A_k. $p \in A_k' \Rightarrow p \in \bar{A}_k \Rightarrow p \in \cup_{i=1}^{n} \bar{A}_i$.

Thus we have shown $\bar{B} \subseteq \cup_{i=1}^{n} \bar{A}_i$ and $\bar{B} \supseteq \cup_{i=1}^{n} \bar{A}_i$, so $\bar{B} = \cup_{i=1}^{n} \bar{A}_i$.

b Again we show that every element $p \in \cup_{i=1}^{\infty} \bar{A}_i$ is in $\bar{B}$. There must be some set $\bar{A}_k$ that contains p. So p is either in A_k or A_k'. If $p \in A_k$, then $p \in \cup_{i=1}^{\infty} A_k = B$. If $p \in A_k'$, then p is a limit point of A_k and every neighborhood $N_r(p)$ contains a point $q \in A_k$. Since $q \in A_k$, $q \in B$ by definition of B, and p is also a limit point of B. So $p \in B' \subseteq \bar{B}$. Thus in either case $p \in \bar{B}$, so $\bar{B} \supseteq \cup_{i=1}^{\infty} A_i$.

This inclusion can be a proper inclusion!

Consider the sets $A_i \subset \mathbb{R}$
$A_i = (1/i, 2)$.

0 is a limit point of the set $\overline{B} = \bigcup_{i=1}^{\infty} A_i$,
but 0 is not in $\bigcup_{i=1}^{\infty} \overline{A_i}$ because it
is not included or a limit point for any
particular set A_i.

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