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18.100B PS3.

1. E is closed. We will show this by showing that E^C is open. Observe that E^C is the union $\{x \in \mathbb{Q} \mid x^2 > 3\} \cup \{x \in \mathbb{Q} \mid x^2 < 2\}$, and call these sets A and B. Consider any point $p \in A$, and let $\delta = \frac{3-p^2}{|p|+3}$, and take $N_\delta(p) = (p-\delta, p+\delta)$. If $p > 0$, then

$$p - \delta = \frac{p^2+3p+3-p^2}{p+3} = \frac{3p+3}{p+3}.$$ Then:

$$(p-\delta)^2 = \left(\frac{3p+3}{p+3}\right)^2 = \frac{9p^2+18p+9}{p^2+6p+9} = 3 + \frac{6(p^2-3)}{(p+3)^2}$$

Since $p^2 > 3$, $(p-\delta)^2 > 3$, so $p-\delta \in A$, and since $\delta > 0$, $p+\delta$ is clearly also in A, so the interval is in A. Similarly, for $p < 0$, we find $(p+\delta)^2 > 3$ and $p-\delta < p$, so both endpoints of the interval must be in A, and so the interval must be contained in A. We have thus found a neighborhood of any point p that is a subset of A, so A is open.

Next, consider $p \in B$ and let $\delta = \frac{2-p^2}{|p|+2}$, we find that $N_\delta(p) = (p-\delta, p+\delta)$ is a subset of B by the same sort of reasoning (tedious details omitted, see also Rudin example 1.1). So B is open. Since $E^C = A \cup B$, and A and B are open, E^C is open $\Rightarrow$ E is closed. ✓

It is easy to see that E is bounded; for example, consider $M = 2$ and $p = 0$; obviously for any $q \in E$, $d(p,q) < M$; otherwise $|q| > 2$ $\Rightarrow q^2 > 4 \Rightarrow q \notin E$. ✓

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E is not compact. To be compact embedded in $\mathbb{Q}$, it must also be compact in $\mathbb{R}$. But the set $\{p \in \mathbb{Q} \mid 2 < p^2 < 3\}$ is not closed in $\mathbb{R}$ (it has limit points at $\sqrt{2}$ and $\sqrt{3}$ but does not contain them) so E is not closed and bounded in $\mathbb{R}$, so it is not compact in $\mathbb{R}$ so it is not compact.

E is open in $\mathbb{Q}$ because a neighborhood can be found for any point $p \in E$ that is a subset of E. Specifically, consider the nbhd of radius $\min\left(\frac{2-p^2}{|p|+2}, \frac{3-p^2}{|p|+3}\right)$. (proof omitted)

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2. Consider the set of points in $\mathbb{R}^k$ w/ only rational coordinates. We know that $\mathbb{Q}$ is countable, so this set is countable because it can be expressed as a k-tuple of $\mathbb{Q}$. (Rudin, thm 2.13 and 2.14). Now we show that any point in $\mathbb{R}^k$ is a limit pt for $\mathbb{Q}^k$. Let $p \in \mathbb{R}^k = \{p_1, p_2, \ldots p_k\}$, and consider some nbhd $N_r(p)$. We can find a point $q \in \mathbb{Q}^k$ that is in $N_r(p)$ for any $r > 0$: for every component p_i, there exists a real number $p_i - r/k$. Because $\mathbb{Q}$ is dense in $\mathbb{R}$, we can find a rational q_i between p_i and $p_i - r/k$ (Rudin, thm 1.20 b). The distance $d(p,q) = \sqrt{\sum_{i=1}^{k}(p_i - q_i)^2}$ is not greater than $\sqrt{\sum_{i=1}^{k}(p_i - (p_i - r/k))^2}$ because of how we defined q_i.
So $d(p,q) \leq \sqrt{\sum_{i=1}^{k}(r/k)^2} = \sqrt{k r^2/k^2} = r/\sqrt{k}$, which is certainly smaller than r, so $q \in N_r(p)$.
Thus any point $p \in \mathbb{R}^k$ is a limit pt for $\mathbb{Q}^k$, so $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. It is a countable dense subset, so $\mathbb{R}^k$ is separable.

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3. Let X be a separable metric space. Then X has a countable dense subset Y. For each element in Y, consider the set of neighborhoods $N_r(y_i)$, $r \in \mathbb{Q}$. This is a countable number of neighborhoods per element of Y, and Y has a countable number of elements, so the union of all these neighborhoods is a union of a countable set of countable sets, which is countable. Call this union $\{V_\alpha\}$.

Now consider an open set $G \subseteq X$ and a point $p \in G$. We know that p is a limit point of Y, since Y is dense in X (or p is a point of Y, in which case it is trivially in at least one neighborhood in $\{V_\alpha\}$). Since G is open, there exists a $r > 0$ such that $N_r(p) \subseteq G$. Since the rationals are dense in the reals, we can find $q \in \mathbb{Q}$ st $0 < q < r$ and note that $N_q(p) \subseteq N_r(p) \subseteq G$. Since p is a limit point of Y, we know that any neighborhood of p, including $N_q(p)$ contains a point $x \in Y$. Since $y \in N_q(p)$, $d(p,y) < q \Rightarrow d(y,p) < q \Rightarrow p \in N_q(x)$. So p is contained in a neighborhood of a point of Y with rational radius - that is, it is contained in some member of $\{V_\alpha\}$. Since x was an arbitrary point of an arbitrary open set G, any arbitrary open set consists of points in sets of $\{V_\alpha\}$. We showed above that $\{V_\alpha\}$ is countable, so X has a countable base.

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4. Given X a metric space where every infinite subset has a limit point, note that X has a countable base, so every open cover is a countable . Suppose X is not compact. Then there is an open cover that has no finite subcover. Write the countable subcover as $\{G_1, G_2, G_3 \ldots\}$. Since no finite collection of these $\{G_n\}$ covers X, $F_n = (G_1 \cup G_2 \ldots \cup G_n)^c$ must be non-empty for any n, but $\bigcap_{n=1}^{\infty} F_n$ is empty. The sets G_n are open by definition, and the sets F_n are finite unions of complements of G_n's, so every F_n is closed. Thus $\bigcap_{n=1}^{\infty} F_n$ is also closed. Also note that the sets F_n are nested such that $F_n \supset F_{n+1}$. Let E be a set consisting of a point from each F_n, and note that E is infinite so it must have a limit point which we will call p. p cannot be in $\bigcap_{n=1}^{\infty} F_n$, since this intersection is empty. Any neighborhood of p contains infinitely many points of E. (Rudin, Thm 2.20). So since E contains only one point from each F_n, every set F_n except finitely many must contain a point in any neighborhood of p. So we can say that p is a limit point of every F_n for n greater than some value k (discarding the finite number that may not have p as a limit pt.) The F_n's are closed, so $p \in F_n$ for $n > k$. But $F_n \subset F_{n+1}$, so p must also be in F_n for $n \leq k$. So $p \in F_n$ for all n. Thus $p \in \bigcap_{n=1}^{\infty} F_n$, which is a contradiction, since we said this intersection was empty, so X must be compact.

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5. a. Let $A \subseteq X$ and $B \subseteq X$ be disjoint closed sets. $A \cap B = \emptyset$ since they are disjoint, and $\bar{A} = A$ and $\bar{B} = B$ since they are closed. So $A \cap \bar{B} = A \cap B = \emptyset$ and $\bar{A} \cap B = A \cap B = \emptyset \Rightarrow$ ✓ A and B are separated.

b. Let $A \in X$ and $B \in X$ be disjoint open sets. Consider a point $p \in A$. We would like to show that $p \notin \bar{B}$. Certainly $p \notin B$ since $p \in A$ and A and B are disjoint. So suppose $p \in B'$. Since $p \in A$ and A is open, p is an interior point and there exists a nbhd $N_r(p) \subseteq A$. But p is a limit point of B' since $p \in B'$. So any nbhd of p, including $N_r(p)$ contains a point $q \in B$. But q is also in A, which is a contradiction since A and B are disjoint. So $p \notin B' \Rightarrow p \notin \bar{B} \Rightarrow A \cap \bar{B} = \emptyset$. Using the same argument with A and B reversed, we find ✓ $\bar{A} \cap B = \emptyset$, so A and B are disjoint.

c. A is a neighborhood of radius δ around p, so A is open. Next we show B is open: Consider any point $q \in B$, and let $d(p,q) = x$. $x > \delta$, so write $x = \delta + \varepsilon$, $\varepsilon > 0$. Thus $d(p,q) = \delta + \varepsilon$. Now consider the neighborhood $N_\varepsilon(q)$, and any point r in this neighborhood. By definition, $d(r,q) < \varepsilon$. From the triangle inequality, $d(p,q) \leq d(p,r) + d(r,q)$

$\Rightarrow \delta + \varepsilon \leq d(p,r) + d(r,q)$ ✓

$\Rightarrow \delta + \varepsilon - d(r,q) \leq d(p,r)$

$\Rightarrow d(p,r) > \delta$, since $d(r,q) < \varepsilon$.

$\Rightarrow r \in B \Rightarrow N_\varepsilon(q) \subseteq B \Rightarrow B$ is open

So A and B are both open. They are clearly disjoint, since $d(p,q)$ cannot be both greater and less than δ, so by part b, they are separable.

d. Let X be a metric space that contains at least the two points x and y. We first show that for some $r \in \mathbb{R}$, $0 < r < d(x,y)$ we can find a point $z \in X$ such that $d(x,z) = r$. Suppose no such point existed. Then we could take the set $A = \{p \in X, d(x,p) < r\}$ and $B = \{p \in X, d(x,p) > r\}$. From c, A and B are separated. We know that $X = A \cup B$ since any point $p \in X$ has $d(x,p) < r$ or $d(x,p) > r$ (we assumed there was no point that has $d(x,p) = r$). But then X is a union of two separated sets $\Rightarrow$ X is not connected, a contradiction. So for any $r \in (0, d(x,y))$ we can find such a point z. There are uncountably many such r's in the interval, so there must be uncountably many points z in X.
$\Rightarrow$ X is uncountable.

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