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18.100b PS4

1. a. Observe that

$$x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n} + 2\sqrt{\alpha}\right) = \frac{1}{2x_n}\left(x_n^2 + 2\sqrt{\alpha}\,x_n + \alpha\right)$$
$$= \frac{1}{2x_n}(x_n + \sqrt{\alpha})^2$$

We can show inductively that x_n is greater than $\sqrt{\alpha}$. The base case is given. Next assume $x_n > \sqrt{\alpha}$. Then $x_n > 0$. $(x_n + \sqrt{\alpha})^2 > 0$ since it is a square. So $\frac{(x_n + \sqrt{\alpha})}{2x_n} > 0 \Rightarrow x_{n+1} - \sqrt{\alpha} > 0$
$\Rightarrow x_{n+1} > \sqrt{\alpha}$. By induction, $x_n > \sqrt{\alpha}$ $\forall n$.

Next we show that $x_{n+1} < x_n$. Note that since $x_n > \sqrt{\alpha}$, $\frac{\alpha}{x_n} < \sqrt{\alpha} < x_n$. So $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$ is less than $\frac{1}{2}(x_n + x_n) = x_n$. That is, $x_{n+1} < x_n$. So $\{x_n\}$ is monotonically decreasing and is bounded below by $\sqrt{\alpha}$, it must therefore converge to some point p.

We will show that $p = \sqrt{\alpha}$. Clearly p is not less than $\sqrt{\alpha}$, since every $x_n > \sqrt{\alpha}$. Suppose $p > \sqrt{\alpha}$. Then $p = \sqrt{\alpha} + \delta$ (for some $\delta > 0$). Then for any $\varepsilon > 0$ we can find a $x_n < p + \varepsilon = \sqrt{\alpha} + \delta + \varepsilon$ (we do not need absolute values since $\{x_n\}$ is monotonic decreasing). Then consider $x_{n+1} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right)$

$\sqrt{\alpha} < \frac{1}{2}\left(\sqrt{\alpha} + \delta + \varepsilon + \frac{\alpha}{[illegible]}\right) < \frac{1}{2}\left(\sqrt{\alpha} + \delta + [illegible] + \frac{\alpha}{[illegible]}\right) = \sqrt{\alpha} + \frac{\delta + \varepsilon}{[illegible]}$

decreasing and converges to p, so p cannot be greater than $\sqrt{\alpha}$. Thus $p = \sqrt{\alpha}$.

b. $\varepsilon_n = x_n - \sqrt{\alpha}$

$\varepsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}\left(x_n + \frac{\alpha}{x_n}\right) - \sqrt{\alpha} = \frac{1}{2x_n}(x_n^2 + \alpha) - \sqrt{\alpha}$

$= \frac{1}{2x_n}(x_n^2 - 2\sqrt{\alpha}\, x_n + \alpha) = \frac{1}{2x_n}(x_n - \sqrt{\alpha})^2$

$= \frac{1}{2x_n} \varepsilon_n^2$.

Since $x_n > \sqrt{\alpha}$ (from part a),

$\varepsilon_{n+1} < \varepsilon_n^2 / 2\sqrt{\alpha}$

Letting $\beta = 2\sqrt{\alpha}$, we show by induction that $\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$.

First, the base case: $\varepsilon_2 < \frac{\varepsilon_1^2}{2\sqrt{\alpha}} = \beta\left(\frac{\varepsilon_1^2}{\beta^2}\right) = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^1}$.

Next, assume $\varepsilon_n < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^{n-1}}$. Then,

$\varepsilon_{n+1} < \frac{\varepsilon_n^2}{\beta} = \frac{1}{\beta}\beta^2\left[\left(\frac{\varepsilon_1}{\beta}\right)^{2^{n-1}}\right]^2 = \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$

So $\varepsilon_{n+1} < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^n}$ for all $n \in \mathbb{N}$.

c. Let $\alpha = 3 \Rightarrow \beta = 2\sqrt{3}$, $x_1 = 2$

$\varepsilon_1 = x_1 - \sqrt{\alpha} = 2 - \sqrt{3} \approx 0.268$

$\frac{\varepsilon_1}{\beta} = \frac{2-\sqrt{3}}{2\sqrt{3}} = \frac{1}{\sqrt{3}} - \frac{1}{2} \approx 0.078 < 1/10$

So $\varepsilon_5 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^4} < 2\sqrt{3}\left(\frac{1}{10}\right)^{2^4} = \frac{2\sqrt{3}}{10^{16}} < 4 \cdot 10^{-16}$

since $2\sqrt{3} < 4$

$\varepsilon_6 < \beta\left(\frac{\varepsilon_1}{\beta}\right)^{2^5} < 2\sqrt{3}\left(\frac{1}{10}\right)^{2^5} = \frac{2\sqrt{3}}{10^{32}} < 4 \cdot 10^{-32}$

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2. Let $\{p_n\}$ and $\{q_n\}$ be Cauchy sequences in a metric space X. Note that $\forall m\ \forall n$,

$$d(p_n, q_n) \leq d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)$$

$$\Rightarrow d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_n, q_m).$$

So we can show that $\{d(p_n, q_n)\}$ is a Cauchy sequence: given some $\varepsilon > 0$ we will show that $\exists N : \forall n \geq N\ \forall m \geq N$ $d(d(p_n, q_n), d(p_m, q_m)) = |d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$. Proof:

Let $\varepsilon > 0$. Since $\{p_n\}$ and $\{q_n\}$ are Cauchy, we can find a N_1 : $\forall n, m \geq N_1$ $d(p_n, p_m) < \varepsilon/2$, and similarly a N_2 : $\forall n, m \geq N_2$ $d(q_n, q_m) < \varepsilon/2$.

So let $N = \max\{N_1, N_2\}$. Then from above:

$$d(p_n, q_n) - d(p_m, q_m) \leq d(p_n, p_m) + d(q_n, q_m)$$

$$\leq \varepsilon/2 + \varepsilon/2 \quad \text{for } m, n > N$$

$$\Rightarrow |d(p_n, q_n) - d(p_m, q_m)| < \varepsilon$$

$$\Rightarrow \{d(p_n, q_n)\} \text{ is Cauchy}$$

Since $\{d(p_n, q_n)\}$ is a Cauchy sequence in $\mathbb{R}^1$, it converges.

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3. a. Denote this equivalence relation as $\sim$.

It is reflexive: $\forall \{p_n\}$ $\lim_{n\to\infty} d(p_n, p_n) = 0$ by the metric axioms

It is symmetric: Suppose $\lim_{n\to\infty} d(p_n, q_n) = 0$.

By the metric axioms $d(p_n, q_n) = d(q_n, p_n)$ so $\lim_{n\to\infty} d(q_n, p_n) = 0$ $\Rightarrow q \sim p$ if $p \sim q$.

It is transitive: Suppose $p \sim q$ and $q \sim r$. Then we show $p \sim r$. Consider $\lim_{n\to\infty} d(p_n, r_n)$. This is less than $\lim_{n\to\infty} d(p_n, q_n) + d(q_n, r_n)$ by the triangle inequality, and both terms go to zero in the limit since $p \sim q$ and $q \sim r$, so $\lim_{n\to\infty} d(p_n, r_n) = 0 \Rightarrow p \sim r$.

So it is an equivalence relation.

b. Given $P, Q \in X^*$, $\{p_n\} \in P$, and $\{q_n\} \in Q$, let $\{p_n'\} \in P$ be another equivalent sequence, and similarly $\{q_n'\} \in Q$. We would like to show that $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p_n', q_n')$. Applying the triangle inequality: $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p_n, p_n') + d(p_n', q_n') + d(q_n', q_n)$

Since $\{p_n\} \sim \{p_n'\}$ and $\{q_n\} \sim \{q_n'\}$, $\lim_{n\to\infty} d(p_n, p_n') = \lim_{n\to\infty} d(q_n, q_n') = 0$ from part a. So $\lim_{n\to\infty} d(p_n, q_n) \leq \lim_{n\to\infty} d(p_n', q_n')$.

Making the same arguments, we find that

$\lim_{n\to\infty} d(p_n', q_n') \leq \lim_{n\to\infty} d(p_n', p_n) + d(p_n, q_n) + d(q_n, q_n') = \lim_{n\to\infty} d(p_n, q_n)$.

So $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(p_n', q_n')$; $\Delta(P, Q)$ is well-defined.

We next show Δ is a metric by checking the axioms:

1) $\Delta(P,P) = \lim_{n\to\infty} d(p_n, p_n) = 0$ ✓

3) $\Delta(P,R) \leq \Delta(P,Q) + \Delta(Q,R)$

$\Delta(P,R) = \lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + d(q_n, r_n)$

by the triangle inequality and

$\leq \Delta(P,Q) + \Delta(Q,R)$ by def.

So Δ is a metric.

c) Consider any Cauchy sequence $\{P_n\}$ in (X^*, Δ). Choose a representative $\{p_{nk}\}$ for each P_n. Since each $\{p_{nk}\}$ is Cauchy, we can find a K_n such that $\forall k,l \geq K_n$, $d(p_{nk}, p_{nl}) < 1/n$. So we can define a sequence $\{q_n\}$ as $q_n = p_{nK_n}$. Consider $d(q_n, q_m)$. By the triangle inequality, $d(q_n, q_m) \leq d(q_n, p_{nk}) + d(p_{nk}, p_{mk}) + d(p_{mk}, q_m)$

$\leq 1/n + \Delta(P_n, P_m) + 1/m$

by our definitions of q_n and Δ.

Therefore $\{q_n\}$ is a Cauchy sequence in (X, d). Given any $\varepsilon > 0$ we can find a N_1 such that $\forall n, m > N_1$ $\frac{1}{n} + 1/m < \varepsilon/2$ and a N_2 such that $\forall n, m > N_2$ $\Delta(P_n, P_m) < \varepsilon/2$ since $\{P_n\}$ is Cauchy in (X^*, Δ). So let $N = \max\{N_1, N_2\}$, then $\forall n, m > N$, $d(q_n, q_m) \leq \frac{1}{n} + \Delta(P_n, P_m) + 1/m < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $\{q_n\}$ is Cauchy. Thus $\{q_n\}$ is in some equivalence class $Q \in X^*$. For any $\varepsilon > 0$ we can find a N such that $\forall n, m \geq N$, $d(q_m, q_n) < \varepsilon$. Then $\forall n \geq N$, $\forall m \geq \max(N, K_n)$

$d(p_n, \ldots) \leq d(p_{\ldots}, q_{\ldots}) + d(\ldots, q_{\ldots}) \ldots$ by triangle ineq.

Thus $\forall n \geq N$, $\Delta(P_n, Q) = \lim_{m\to\infty} d(p_{nm}, q_m)$

$\leq 1/n + \varepsilon$

(from above, since m is arbitrarily large).

So P_n converges to Q in (X^*, Δ): for any $\varepsilon > 0$ we can find an N sufficiently large that $\forall n \geq N$, $\frac{1}{n} < \varepsilon/2$ and $\forall n,m \geq N$ $d(q_n, q_m) < \varepsilon/2$.
Then $\Delta(P_n, Q) < 1/n + \varepsilon/2 < \varepsilon/2 + \varepsilon/2 = \varepsilon$.
So for any ε we can find a N such that $\forall n \geq N$ $\Delta(P_n, Q) \leq \varepsilon \Rightarrow P_n$ converges to Q in $(X^*; \Delta)$

d. Let $p \in X$ and $P_p \in X^*$ such that P_p contains a sequence $\{p_n\}$ where every term is p, and likewise $q \in X$, $\{q_n\} \in P_q \in X^*$. Then

$\Delta(P_p, P_q) = \lim_{n\to\infty} d(p_n, q_n)$ by definition of Δ (since we can choose any representative sequences from P_p and P_q)

$= \lim_{n\to\infty} d(p, q)$ by def of p_n and q_n

$= d(p, q)$

e. Any point in X^* is either a point in $\varphi(X)$ or a limit point of $\varphi(X)$. Consider $P \in X^*$.
If P's equivalence class contains a constant sequence, then that constant sequence is $\varphi(q)$ for some $q \in X$. Then $P \in \varphi(X)$; otherwise

P. Given any $\varepsilon > 0$ we can find a N such that $\forall m, n \geq N$ $d(P_m, P_n) < \varepsilon$ since $\{P_n\}$ is Cauchy. So take some $k \geq N$ and construct the constant sequence $P_{P_k} = \{P_k, P_k, P_k \dots\}$

Then $\Delta(P, P_{P_k}) = \lim_{n\to\infty} d(P_n, P_{P_{k_n}}) = \lim_{n\to\infty} d(P_n, P_k)$

In the limit, when $n \geq N$, this is less than ε (as shown above). So any point in X^* is either a point or a limit point of $\varphi(x)$, $\varphi(x)$ is dense in X^*

If X is complete, then $\varphi(X) = X$. First, note that for any $p \in X$, $\varphi(p)$ is a constant sequence, so it is a Cauchy sequence so $\varphi(p) \in X^*$. Thus $\varphi(X) \subseteq X^*$. Next consider some element $P \in X^*$. It can be represented by some Cauchy sequence in X, $\{P_n\}$. Since X is complete, $\{P_n\}$ converges to $q \in X$. Then consider the Cauchy sequence with all terms equal to q, P_q. Since $\{P_n\}$ converges to q, $\lim_{n\to\infty} d(P_n, P_{q_n}) = \lim_{n\to\infty} d(p_n, q) = 0 \Rightarrow$

P_n is in the same equivalence class as P_q. Since $P_q \in \varphi(X)$ and P_q is a representation of P and P was an arbitrary point in X^*, $X^* \subseteq \varphi(X)$

So $X^* = \varphi(X)$ if X is complete.

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4. a. d_p satisfies the metric axioms:

1) If $x \neq y$, $d(x,y) = \frac{1}{p^{t(x-y)}}$. Since $t(x-y)$ is an integer, $\frac{1}{p^{t(x-y)}} > 0 \Rightarrow d(x,y) > 0$.
If $x = y$, $x - y = 0$ and $d(x,y) = |0|_p = 0$

2) Note that $t(n) = t(-n)$: if $n = p^{t(n)} m$, $-n = p^{t(n)}(-1 \cdot m)$ since -1 is not divisible by p.
Thus $d(x,y) = |x-y|_p = p^{-t(x-y)} \Rightarrow p^{-t(-(x-y))} = p^{-t(y-x)}$
$= |y-x|_p = d(y,x)$

3) Note that for integers a, b, $t(a+b) >= \min\{t(a), t(b)\}$ because they can be written $a = p^{t(a)} m$ and $b = p^{t(b)} n$ for m, n that do not divide p. Assume w/o loss of generality that $t(a) >= t(b)$. Then $a = p^{t(b)} p^{t(a)-t(b)} m$, and $a+b = p^{t(b)}[p^{t(a)-t(b)}m + n]$
By our assumption, $t(a)-t(b)$ is non-negative, so $t(a,b) > t(b)$. If $t(a) < t(b)$, then we simply swap a and b, that is, $t(a+b) >= \min\{t(a); t(b)\}$.
Also note that, trivially, $t(ab) = t(a) + t(b)$
Next consider $x, y \in \mathbb{Q}$. Write $x = a/b$, $y = c/d$.
Then $x+y = \frac{ad+bc}{bd}$

$t(x+y) = t(ad+bc) - t(bd) = t(ad+bc) - t(b) - t(d)$

$>= \min\{t(ad), t(bc)\} - t(b) - t(d)$

$>= \min\{t(a)+t(d), t(b)+t(c)\} - t(b) - t(d)$

$>= \min\{t(a) - t(b), t(c) - t(d)\}$

So $|x+y|_p = \frac{1}{p^{t(x+y)}} <= \max\{t(x), t(y)\}$ since the inequality is reversed when the $p^{t(x+y)}$ is moved to the denominator (maximizing the denominator minimizes the function).

$d_p(x,y) = |x-y|_p = |(x-z)+(z-y)|_p$
$<= \max\{|x-z|_p, |z-y|_p\}$ from above
$= \max\{d_p(x,z), d_p(z,y)\}$ by def

So d_p satisfies the ultrametric triangle inequality; it clearly also satisfies the normal inequality
$d_p(x,y) <= d_p(x,z) + d_p(z,y)$ ✓

d_p satisfies the 3 metric axioms, so it is a metric.

b The sequence $\{p_n\} = \{2^n+1\}_{n=1}^{\infty}$ converges in $(\mathbb{Q}, d_p)$ to 1. For any ε, we can find a N such that $\forall n > N, d_2(p_n, 1) < \varepsilon \Rightarrow |p_n - 1|_2 < \varepsilon \Rightarrow |2^n+1-1|_2 < \varepsilon$
$\Rightarrow |2^n|_2 < \varepsilon \Rightarrow \frac{1}{2^n} < \varepsilon \Rightarrow 1 < 2^n\varepsilon \Rightarrow 2^n > 1/\varepsilon$.
So we simply need to choose N large enough that $2^N > 1/\varepsilon$, i.e. $N > \log_2(1/\varepsilon)$. With this N, $\forall n > N$ $d(p_n, 1) < \varepsilon$.
So $\lim_{n\to\infty} p_n = 1$ ✓

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