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	Problem Set 5	
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Problem 1:

Part a Since $\sum a_n$ diverges, there are two possibilities: either $\lim_{x\to\infty} a_n = 0$ or $\lim_{x\to\infty} \neq 0$. In the latter case, $a_n \to p$, for some extended real number $p \neq 0$. Then $\frac{a_n}{1+a_n} \to \frac{p}{1+p}$, which is clearly non-zero. So by the divergence test, $\sum \frac{a_n}{1+a_n}$ diverges. Otherwise, $a_n \to 0$. In this case, we can thus find a $N > 0$ such that $\forall n \geq N$, $a_n < 1$ by the definition of convergence. Then $2(\frac{a_n}{1+a_n}) < 2(\frac{a_n}{2}) = a_n$. Thus, by the comparison test, $\sum 2(\frac{a_n}{1+a_n})$ diverges since $\sum a_n$ diverges. Hence, $\sum \frac{a_n}{1+a_n}$ diverges. Thus $\sum \frac{a_n}{1+a_n}$ diverges for any a_n where $\sum a_n$ diverges. ✓

Part b We know $\forall n a_n > 0$, so $s_{n+1} > s_n$ (s_n is monotonically increasing. So:

$$\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq \frac{a_{N+1} + ...a_{N+k}}{s_{N+k}} = \frac{s_{N+k} - s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

We can use this to show that $\sum \frac{a_n}{s_n}$ is divergent. Let S_n be the nth partial sum of this sequence: $S_n = \sum_{i=1} n \frac{a_i}{s_i}$. Then we consider the difference between two partial sums S_{N+k} and S_N:

$$S_{N+k} - S_N = \frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$ ✓

Since $\sum a_n$ diverges, $\lim_{n\to\infty} s_n = \infty$, and if we fix N, $\lim_{k\to\infty} \frac{s_N}{s_{N+k}} = 0$. So $\lim_{k\to\infty} 1 - \frac{s_N}{s_{N+k}} = 1$. Combined with the inequality above, this means that, for any N we can find a k such that the difference between the $N+k$th and Nth partial sums is at least $1-\epsilon$, for any $\epsilon > 0$. Arbitrarily choosing $\epsilon = \frac{1}{2}$, we can find a k for any N such that $S_{N+k} - S_N \geq \frac{1}{2}$. This means that the sequence of partial sums S_n does not satisfy the Cauchy criterion, so it does not converge, and thus $\sum \frac{a_n}{s_n}$ diverges.

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Part c

$$\begin{aligned}
\frac{a_n}{s_n^2} &= \frac{a_n}{s_n s_n} \\
&\leq \frac{a_n}{s_n s_{n-1}} \text{ by the monotonicity of } s_n \text{ as shown above} \\
&= \frac{s_n - s_{n-1}}{s_n s_{n-1}} \text{ by definition of } s_i \\
&= \frac{s_n}{s_n s_{n-1}} - \frac{s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}
\end{aligned}$$ ✓

We use this to show that $\sum \frac{a_n}{s_n^2}$ converges. Using the inequality above, this is less than the series $\sum_{n=2}^{\infty} \frac{1}{s_{n-1}} - \frac{1}{s_n}$. Consider the partial sums $S_N = \sum_{n=2}^{N} \frac{1}{s_{n-1}} - \frac{1}{s_n} = \frac{1}{s_1} - \frac{1}{s_2} + \frac{1}{s_2} - \frac{1}{s_3} + ... + \frac{1}{s_{N-1}} - \frac{1}{s_N}$. This is a telescoping sum which cancels to $\frac{1}{s_1} - \frac{1}{s_N} = \frac{1}{a_1} - \frac{1}{s_N}$. We know $\frac{1}{a_1}$ is constant, and $s_n \to \infty$ because a_n diverges, so $\lim_{N\to\infty} S_N = \frac{1}{a_1}$. Thus the sequence of partial sums converges, so we have shown that $\sum \frac{a_n}{s_n^2}$ is less than a series that converges, so it converges by the comparison test.

Part d Nothing can be said about the convergence of the first series. We will show an example of an appropriate series a_n that makes $\sum \frac{a_n}{1+na_n}$ diverge, and another choice for a_n that makes it converge. First, consider $\sum \frac{a_n}{1+na_n}$. If $a_n = 1$ (clearly $\sum a_n$ is divergent), the series becomes $\sum \frac{1}{1+n}$. Each term is larger than $\frac{1}{2n}$ (for $n > 2$), which diverges (since it is a multiple of $\frac{1}{n}$), so by the comparison test $\sum \frac{a_n}{1+na_n}$ diverges.

Next consider a sequence a_n that equals 1 when n is a perfect square and $\frac{1}{n^2}$ otherwise. Then $\sum a_n$ diverges because its subseries $\sum_{\{n \text{ a perfect square}\}} 1$ certainly diverges. We will show, however, that $\sum \frac{a_n}{1+na_n}$ converges. We can write this series as $\sum b_n + \sum_{n\text{a perfect square}} \frac{1}{1+n}$, where $b_n = \frac{\frac{1}{n^2}}{1+n\frac{1}{n^2}} = \frac{1}{1+n^2}$ when n is not a perfect square and 0 when it is. By the comparison test, $\sum b_n$ converges because $\forall n\ b_n \leq \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges. And we can write $\sum_{\{n \text{ a perfect square}\}} \frac{1}{1+n}$ as $\sum_{m=1}^{\infty} \frac{1}{1+m^2}$ (letting $n = m^2$). This series converges, since every term is less than $1/m^2$, which is a convergent series. Thus $\sum \frac{a_n}{1+na_n}$ is a sum of two convergent series and converges itself for this choice of a_n. ✓

We show that $\sum \frac{a_n}{1+n^2a_n}$ always converges. Note that $\frac{a_n}{1+n^2a_n} = \frac{1}{\frac{1}{a_n}+n^2}$. Since $\forall n\ a_n > 0$ by definition, the denominator is always greater than n^2, and so the terms of the series are less than those of the series $\frac{1}{n^2}$, which we know converges. So by the comparison test, $\sum \frac{a_n}{1+n^2a_n}$ always converges for any such a_n. ✓

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Problem 2:

Let f be a continuous function from a metric space X to a metric space Y, and E be a subset of X. For any $p \in E$, clearly $f(p) \in f(E) \subseteq \overline{f(E)}$. Now consider $p \in E'$. Then p is a limit point of E. We will show that $f(p)$ is a limit point of $f(E)$. That is, given any $\epsilon > 0$, we will find a point $q \in E$ such that $d_Y(f(p), f(q)) < \epsilon$. Since f is continuous, there exists a δ such that $\forall x \in X$ where $d_X(p, x) < \delta$, $d_Y(f(p), f(x)) < \epsilon$. Since p is a limit point of E, we can find a point $q \in E$ such that $d_X(p, q) < \delta$ and thus $d_Y(f(p), f(q)) < \epsilon$. So $f(p)$ is a limit point of $f(E)$. Thus, $\forall p \in E'$, $f(p) \in \overline{f(E)}$

We have shown that $f(E) \subseteq \overline{f(E)}$ and that $f(E') \subseteq \overline{f(E)}$. So since $\overline{E} = E \cup E'$, $f(\overline{E}) \subseteq \overline{f(E)}$. ✓

As an example, consider f from R to R such that $f(x) = \frac{1}{x}$. Let $E = 1, 2, 3, 4....$ Clearly, E has no limit points. But $f(e) = \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, ...$, which has a limit point at zero. 0 is in $\overline{f(E)}$ but not in $f(\overline{E})$, so $f(\overline{E}) \subset \overline{f(E)}$.

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Problem 3:

Let f and g be continuous mappings from a metric space X to a metric Y, and E be a dense subset of X. This means that $\overline{E} = X$. From problem 2, $f(\overline{E}) \subseteq \overline{f(E)}$. Thus, $f(X) = f(\overline{E}) \supseteq \overline{f(E)}$. Thus any point in $f(X)$ is in the closure of $f(E)$, so it is either a point of $f(E)$ or a limit point of $f(E)$. By definition of density, $f(X)$ is dense in $f(E)$.

Now suppose that $\forall p \in E\ g(p) = f(p)$. We will show by contradiction that $\forall p \in X\ g(p) = f(p)$. Suppose there is some point $q \in X$ such that $g(q) \neq f(q)$ (it must be the case that $q \notin E$). Because E is dense in X, q is a limit point of E. So we can construct a sequence $\{p_1, p_2, p_3, ...\}$ of elements of E such that the sequence $\{p_i\}$ converges to q. Because f is continuous, the sequence $\{f(p_i)\}$ converges to $f(q)$ since $\{p_i\}$ converges to q. Because each p_i is in E, $\forall i\ f(p_i) = g(p_i)$; thus, $\{g(p_i)\}$ must also converge to $f(q)$ (two convergent sequences cannot converge to different points). But $f(q) \neq g(q)$, so $\{g(p_i)\}$ does not converge to $g(q)$, contradicting the continuity of g. Therefore, there can be no such point q and $\forall p \in X\ g(p) = f(p)$.

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Problem 4:

First, note that if E is closed, then it is easy to show that $f(E)$ is bounded. In this case, E is a closed and bounded subset of R, so E is compact. $f(E)$ is uniformly continuous, so it is continuous, and it is defined on E, so $f(E)$ is therefore also compact. Compactness implies closed and boundedness, so $f(E)$ must be bounded.

We will now show that this is also true in other cases. Let E be a bounded set in R. There is thus some number $M \in R$ such that $\forall x \in E\ |x| < M$. We first show that the closure of E is also bounded. Consider a limit point p of E. Because p is a limit point, for every $\epsilon > 0$ there exists a point q in E such that $|p - q| < \epsilon$. Using the triangle inequality between p, q, and 0, $|q| \leq |p - q| + |p| < \epsilon + M$. Since we can make ϵ arbitrarily small, for any $K > M$, $\forall x \in \overline{E}\ |x| < K$. So $\overline{E}$ is bounded. It is also closed by definition, and it is a subset of R, so it is compact.

Let f be a uniformly continuous function from E to R. By definition of uniform continuity, for every $\epsilon > 0$ we can find a $\delta > 0$ such that $\forall p \in E\ \forall q \in E$ such that $|p - q| < \delta$, $|f(p) - f(q)| < \epsilon$. We arbitrarily choose 1 as a positive value for ϵ and use the uniform continuity of f to obtain the corresponding value δ such that $\forall p \in E\ \forall q \in E$ such that $|p - q| < \delta$, $|f(p) - f(q)| < 1$. Next we consider the neighborhoods of radius δ around every point $x \in E$: $N_\delta(x) = \{p \in E \mid |x - p| < \delta\}$. Certainly every point of E is in one of these neighborhoods. Also, every limit point x of E is contained in one of these neighborhoods: since x is a limit point, we can find a point $q \in E$ such that $|x - q| < \delta$, and so $x \in N_\delta(q)$. Since the neighborhoods are open sets, their union is an open set, and so because $\bigcup_{x \in E} N_\delta(x)$ contains $\overline{E}$, it is an open cover of $\overline{E}$. $\overline{E}$ is compact, so the open cover contains a finite subcover: a set of points $\{x_1, x_2, ..., x_n\}$ such that $\bigcup_{i=1}^{n} N_\delta(x_i)$ contains $\overline{E}$. It clearly also contains E.

Since each point x_i is in E, $f(x_i)$ is defined. Note that every point of E is within δ of one of the x_i's, since E is contained in their union of radius-δ neighborhoods. So because of our choice of δ based on uniform continuity, $\forall x \in E, \exists i$ such that $|f(x) - f(x_i)| < 1$. Since there are a finite number of points x_i, we can find $\max\{|f(x_i)|\}$. Now we observe that for every x in E, $|f(x)|$ cannot be less than (by the triangle inequality) the distance from $f(x)$ to the nearest $f(x_i)$, which we just showed was less than 1, plus $|f(x_i)|$, which cannot be greater than the maximum value $\max\{|f(x_i)|\}$. So $\forall x \in E, |f(x)| < 1 + \max\{|x_i|\}$, which means $f(x)$ is bounded.

We give an example that shows that this does not apply if E is unbounded. Consider the function $f(x) = \sqrt{x}$, defined on the interval of $[0, \infty)$. We will show that $f(x)$ is uniformly continuous. Given some $\epsilon > 0$, let $\delta = \epsilon^2$. Then, for any numbers x and y that satisfy $|y - x| < \delta$, assume without loss of generality that $y > x$. Then $f(y) - f(x) = \sqrt{y} - \sqrt{x} < \sqrt{x + \delta} - \sqrt{y} \leq \sqrt{x} + \sqrt{\delta} - \sqrt{x} = \sqrt{\delta} = \epsilon$. So $f(x)$ is uniformly continuous on this domain. Also note that $f(x) = \sqrt{x}$ is obviously unbounded on this infinite interval.

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Problem 5:

We will first show that $\lim_{t\to x} f(t) = 0$ for every real number x. Let $\{p_1, p_2, p_3, ...\}$ be a sequence of real numbers that converges to x. We will show that $\{f(p_n)\}$ converges to $f(x)$. For every n such that p_n is irrational, $f(p_n) = 0$ by definition of the function. Therefore, for any subsequence $\{p_{q_n}\}$ of irrational numbers in $\{p_n\}$, the corresponding subsequence $\{f(p_{q_n})\}$ of $\{f(p_n)\}$ will converge to 0. Therefore, we will have shown $\{f(p_n)\}$ converges to 0 if we can verify that for every sequence of rational numbers $\{a_n\}$ that converges to x, the sequence $\{f(a_n)\}$ converges to 0.

So we consider a sequence $\{a_n\}$ of rational numbers (write them as a ratio of two integers in lowest terms) that converges to x. We know such a convergent sequence exists because the rationals are dense in the reals. We will show that the denominators must approach infinity. Consider the set $\{b_n\}$ of rational numbers in the interval $(x - a_1, x + a_1)$ with (lowest-terms) denominator not greater than some arbitrary integer k. Because the denominator is not greater than k, the distance between any two elements in the sequence is not less than $\frac{1}{k}$. The interval has finite length (specifically, length $2a_1$), and the elements are spaced some minimum distance ($\frac{1}{k}$) apart, so there can only be a finite number of elements in the set (at most $2a_1k$ elements). Thus, we can find a the minimum distance $\epsilon = \min\{|x - b_n|$ such that no element in $\{b_n\}$ is within ϵ of x. We said earlier that $\{a_n\}$ was a sequence that converges to x, so we must be able to find an integer N such that all elements in the sequence after a_N are within ϵ of x. Since all elements in $\{b_n\}$ are at least ϵ away, $\forall m > N\ a_m \notin \{b_n\}$. $\{b_n\}$ contains all elements closer to x than a_1 with denominator less than k, so a_{N+1} (and all following elements) must have a denominator larger than k. Since k was arbitrary, we have shown that the denominators of $\{a_n\}$ approach infinity. By definition of f, this means that $\{f(a_n)\}$, which is $\frac{1}{\text{denominator}}$ must go to zero.

Thus, for any $\{a_n\}$ composed of rational numbers that converges to x, $\{f(a_n)\} \to 0$. We showed earlier that we can merge any sequence of irrational numbers into the rational sequence (since the irrationals correspond to a 0 value of f) and still converge to zero. So for any sequence of real numbers $\{p_n\}$ that converges to some real x, $\{f(p_n)\}$ converges to zero.

We use this to show that $f(x)$ is continuous at every irrational point and has a simple discontinuity at every rational point. Suppose x is irrational. Then $f(x) = 0$ by definition of the function, and $\lim_{t\to x} f(t) = 0 = f(x)$, so f is continuous at x by definition of continuity. Similarly, if x is rational, then $f(x) \neq 0$, because $f(x)$ is the multiplicative inverse of the lowest terms denominator of x, which cannot be zero. But the limit at x still exists and is zero: $\lim_{t\to x} f(t) = 0 \neq f(x)$. This is the definition of a discontinuity of the first kind.

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