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Problem Set 6

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Problem 1: Let f be a continuous mapping from the closed unit interval $I = [0, 1]$ into I. We will show that there exists some $x \in I$ where $f(x) = x$.

First, note that if $f(0) = 0$ or $f(1) = 1$, then we are trivially done. Otherwise, the range of f requires that $f(0) > 0$ and $f(1) < 1$. Consider $g(x) = x - f(x)$. Then $g(0) = 0 - f(0) < 0$ and $g(1) = 1 - f(1) > 0$, applying the previous inequalities. We know that g is continuous because it is a linear combination of $f(x)$ and x, which are both continuous. Since $g(0) < 0 < g(1)$, the intermediate value theorem indicates that there exists some $x \in (0, 1)$ such that $g(x) = 0$. At this point x, $g(x) = x - f(x) = 0$. Therefore, $f(x) = x$ at this point x. √ 10/10

Problem 2: Let f be a function defined on (a, b) such that $f'(x) > 0$. We show by contradiction that f is strictly increasing. Suppose it is not. Then there exists some $x, y \in (a, b)$ such that $x < y$ but $f(x) \geq f(y)$. The mean value theorem tells us that there is some point $p \in (x, y) \subseteq (a, b)$ such that $f'(p) = \frac{f(y)-f(x)}{y-x}$. But this fraction is either zero or negative, so $f'(p) \leq 0$, contradicting the assumption that $\forall x \in (a, b)\ f'(x) > 0$.

Now let $g(x) = f^{-1}(x)$; that is, g is defined by $g(f(x)) = x$. We will first show that g is continuous. Consider any point y in the domain of g. Then there is some $x \in (a, b)$ such that $y = f(x)$ (and thus $g(y) = g(f(x)) = x$).Since (a, b) is an open interval that contains x, we can choose points p, q such that $a < p < x < q < b$. Then $[p, q]$ is a compact set (closed and bounded in $\mathbb{R}$) that contains x. Applying Theorem 4.17, $g = f^{-1}$ is continuous on the image of $[p, q]$, which contains y. Thus g is continuous at y. Since y was an arbitrary point in g's domain, g is everywhere continuous.

We will next show that g is differentiable and that $g'(f(x)) = \frac{1}{f'(x)}$. Take $\{y_n\}$ to be a sequence in the image of (a, b) under f that converges to a point y also in that image; we also require that no element y_n actually equals y. Then let $x = g(y)$ (and hence $y = f(x)$). By the continuity of g, the sequence $\{x_n\} = \{g(y_n)\}$ converges to $g(y) = x$. Now we consider the sequence of difference quotients:

$$\frac{g(y_n) - g(y)}{y_n - y} = \frac{x_n - x}{f(x_n) - f(x)} = \frac{1}{\frac{f(x_n)-f(x)}{x_n - x}}$$

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Since $x_n \to x$, this sequence converges to $\frac{1}{f'(x)}$ by definition of the derivative of f. We have thus shown (by the sequential definition of a limit) that $\lim_{t \to x} \frac{f(t)-f(x)}{t-x} = \frac{1}{f'(x)}$, so by definition $g'(x) = \frac{1}{f'(x)}$.

Problem 3: Consider the function $f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$. Note that every term contains at least one power of x, so $f(0) = 0$. Also note that $f(1) = C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1}$ which we are given equals zero. Because f is a polynomial function, it is continuous and differentiable. Differentiating f, we obtain: $f'(x) = C_0 + C_1 x + \cdots + C_{n-1}x^{n-1} + C_n x^n$. Observe that the equation we are given is $f'(x) = 0$. By the mean value theorem, there exists some point $x \in (0, 1)$ where $f'(x) = 0$. Therefore, the equation has some real root between 0 and 1.

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Problem 4: Let f be a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 be the suprema of $|f(x)|$, $|f'(x)|$, $|f''(x)|$ respectively. We will show that $M_1^2 \leq 4M_0M_2$.

Consider the second Taylor polynomial $P_2(t) = f(t) + \frac{f'(t)}{2}x$. Let $h = \sqrt{\frac{M_0}{M_2}}$. For some $x \in (a, \infty)$, we apply Taylor's theorem on the interval $(x, x+2h) \subset (a, \infty)$, and find that for some $\xi \in (x, x+2h)$,

$$\begin{aligned}
f(x+2h) &= P(x+2h) + \frac{f''(\xi)}{2}(x+2h-x)^2 \\
&= f(x) + f'(x)2h + \frac{f''(\xi)}{2}(2x)^2 \\
f'(x)2h &= f(x+2h) - f(x) - 2f''(\xi)h^2 \\
f'(x) &= \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)
\end{aligned}$$

Since $M_0 = \sup|f(x)|$, $|f(x+2h) - f(x)| \leq 2M_0$; also $f''(\xi) \leq \sup|f''(x)| = M_2$. Hence,

$$\begin{aligned}
|f'(x)| &\leq \frac{M_0}{h} + hM_2 \\
&= M_0\sqrt{\frac{M_2}{M_0}} + M_2\sqrt{\frac{M_0}{M_2}} = \sqrt{M_0M_2} + \sqrt{M_0M_2} = 2\sqrt{M_0M_2} \\
|f'(x)|^2 &\leq 4M_0M_2 \\
M_1^2 &\leq 4M_0M_2 \qquad \checkmark
\end{aligned}$$

We now show an example where $M_1^2 = 4M_0M_2$. With $a = -1$, let

$$f(x) = \begin{cases} 2x^2 - 1 & (-1 < x < 0) \\ \frac{x^2-1}{x^2+1} & (0 \leq x < \infty) \end{cases}$$

Differentiating each piece separately,

$$f'(x) = \begin{cases} 4x & (-1 < x < 0) \\ \frac{4x}{(x^2+1)^2} & (0 \leq x < \infty) \end{cases} \qquad f''(x) = \begin{cases} 4 & (-1 < x < 0) \\ \frac{-4(3x^2-1)}{(x^2+1)^3} & (0 \leq x < \infty) \end{cases}$$

Note that $2x^2 - 1$ and $\frac{x^2-1}{x^2+1}$ are both always less than 1. $\lim_{x\to-1} 2x^2 - 1 = \lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$, so $M_0 = 1$. For M_1, $\sup_{(-1,0)} |4x| = 4$ since $\lim_{x\to-1} 4x = -4$, and it is easy to show that $\frac{4x}{(x^2+1)^2}$ is everywhere less than 4, so $M_1 = \sup|f'(x)| = 4$. For M_2, clearly $\sup_{(-1,0)} |4| = 4$, and $\left|\frac{-4(3x^2-1)}{(x^2+1)^3}\right|$ is also everywhere less than 4, so $M_2 = \sup|f''(x)| = 4$. So $M_1^2 = 4^2 = 16 = 4\ 4\ 1 = 4M_0M_2$. The inequality can in fact be equal.

This argument does not necessarily hold if the interval (a, ∞) is replaced with $(0, 1)$. To obtain the inequality, we required that the function (and M_0, M_1, M_2) be defined on the interval $[x, x+2h]$. Since $h = \sqrt{\frac{M_0}{M_2}}$, if, say, $M_0 > M_2$, then $h > 1$ and there can be no interval $[x, x+2h]$ contained in $(0, 1)$.

example? $\frac{9}{10}$

Problem 5: Let f be a function differentiable on $[a, b]$ such that for some $A \in \mathbb{R}$, $|f'(x)| \leq A|f(x)|$ on $[a, b]$. We will show that $\forall x \in [a, b]\ f(x) = 0$.

Consider first the case when $b - a < \frac{1}{A}$. Let $M_0 = \sup|f(x)|$ and $M_1 = \sup|f'(x)|$. Note that, since $|f'(x)| \leq A|f(x)|$, $M_1 \leq AM_0$. We will show that $\forall x \in [a,b]\ |f(x)| \leq M_1(b-a)$. Suppose this is not true. Then there exists some point $p \in [a,b]$ such that $|f(p)| > M_1(b-a)$. Then:

$$\begin{aligned}
|f(p)| &> M_1(b-a) \\
\frac{|f(p)|}{b-a} &> M_1 \\
\frac{|f(p)| - |f(a)|}{b-a} &> M_1 \text{ since } f(a) = 0 \\
\frac{|f(p) - f(a)|}{|b-a|} &> M_1 \\
\frac{|f(p) - f(a)|}{|p-a|} &\geq \frac{|f(p) - f(a)|}{|b-a|} > M_1 \text{ since } p \leq b
\end{aligned}$$

By the mean value theorem, we find some point $q \in [a,p] \subseteq [a,b]$ with $f'(q) = \frac{f(p)-f(a)}{p-a}$. Then $|f'(q)| = \frac{|f(p)-f(a)|}{|p-a|} > M_1$, contradicting our definition of M_1 as $\sup|f'(x)|$. Therefore, no such point p can exist and $\forall x \in [a,b]\ |f(x)| \leq M_1(b-a)$.

Applying the inequality between M_0 and M_1, we obtain $\forall x \in [a,b]\ |f(x)| \leq M_1(b-a) \leq AM_0(b-a)$. Because $|f(x)|$ is a continous function defined on the compact set $[a,b]$, it achieves its supremum M_0 at some point. So $M_0 \leq A(b-a)M_0$. Since we have restricted ourselves to cases where $b-a < \frac{1}{A}$, $A(b-a) < 1$. M_0 cannot be greater than zero, because if it were, then $M_0 > A(b-a)M_0$, contradicting the inequality above. M_0 cannot be negative by definition, so $M_0 = 0$. Therefore, $|f(x)| = 0$ on every point in the interval $[a,b]$: if $|f(p)|$ were non-zero at some point $p \in [a,b]$, it would contradict our definition of M_0. So $f(x) = 0$ for every $x \in [a,b]$.

We now generalize this result to the general case in which the interval $[a,b]$ has arbitrary length. Let $K = \frac{1}{2A}$. We will use induction (on n), dividing $[a,b]$ into successive intervals of length K and showing that $f(x) = 0$ on those intervals. Let the induction hypothesis be that $f(x) = 0$ for every point on the interval $[a+nK, \min\{a+(n+1)K\ ,\ b\}]$ (we must use the minimum in that expression to account for the final subinterval, where $a+(n+1)K$ may be greater than b). The base case $n = 0$ follows from our previous result, since $f(a) = 0$ is given and K was chosen so $K < \frac{1}{A}$. We assume the inductive hypothesis is true for $n-1$ and show it is then also true for n. By the inductive hypothesis, we know $f(a+nK) = 0$. We then consider the interval $[a+nK, \min\{a+(n+1)K, b\}]$. The interval has length $\min\{a+(n+1)K, b\} - (a+nK) \leq a+(n+1)K - (a+nK) = K < \frac{1}{A}$, so we can apply our previous result and show that $f(x) = 0$ on the interval. This proves the induction hypothesis for n. By induction, therefore, the induction hypothesis is true for all n. Thus $f(x) = 0$ on the entire interval $[a,b]$.

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