18.100b **Dan Ports**

Problem Set 7

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2003/04/25 drkp@mit.edu

Problem 1: Let f be a continuous function defined on $[a, b]$ such that $f \geq 0$ and $\int_a^b f(x)\, dx = 0$. We show by contradiction that $f(x) = 0$ for all $x \in [a, b]$. Suppose that there exists some point $p \in (a, b)$ such that $f(p) \neq 0$. (Note that it suffices to consider only the interior (a, b) because if $f(p) \neq 0$ for one of the endpoints of the interval, then $f(p)$ must also be non-zero for some interior point by continuity). Since $f \geq 0$ on the entire interval, $f(p) > 0$. We can use the continuity of f to find a value $\delta > 0$ such that $f(x) > \frac{p}{2}$ for every point within δ of p. We then define $\delta' = \min\{\delta, p - a, b - p\}$ (this handles the cases in which $p - \delta < a$ or $p + \delta > b$). Clearly $f(x) \geq \frac{p}{2}$ on the interval $[p - \delta', p + \delta'] \subseteq [a, b]$.

We use this δ' to define a partition $P = \{a, p - \delta', p + \delta', b\}$. We show that the lower Riemann sum $L(P, f)$ is greater than zero. Since $f \geq 0$ on $[a, b]$, the inf f over the sub-intervals $[a, p - \delta']$ and $[p + \delta', b]$ are non-negative, and thus they do not contribute negatively to the lower Riemann sum, so $L(P, f)$ is bounded below by the component from the $[p - \delta', p + \delta']$ interval:

$$L(P, f) \geq 2\delta' \inf_{\{p - \delta' \leq x \leq p + \delta'\}} f(x) > 2\delta' \frac{p}{2} > 0$$

The integral over the interval $[a, b]$ is bounded below by this: √ 10/10

$$\int_a^b f(x)\, dx = \underline{\int_a^b} f(x) dx \geq L(P, f) > 0$$

We have just shown that $\int_a^b f(x)\, dx$ is greater than zero, contradicting our assumption that it is zero. Thus no such p can exist; $f(p) = 0$ for all $x \in [a, b]$.

Problem 2: Let f be a monotonically decreasing function defined on $[1, \infty)$ such that $\forall x\, f(x) \geq 0$. We will show that $\int_1^\infty f(x)\, dx$ converges if and only if $\sum_{n=1}^\infty f(n)$ converges. Note that since f is monotonically decreasing, $f \in \mathscr{R}$ on $[1, b]$ for any $b > 1$.

Suppose the series $\sum_{n=1}^\infty f(n)$ converges. We will show that $\int_1^\infty f(x)\, dx = \lim_{n\to\infty} \int_1^n f(x)\, dx$ converges. Let n be an integer greater than 1, and define a partition P_n of the interval $[1, n]$ to be the integers $\{1, 2, \ldots, n\}$. Since f is monotonically decreasing, $\sup_{\{i \leq x \leq i+1\}} f(x) = f(i)$ and $\inf_{\{i \leq x \leq i+1\}} f(x) = f(i + 1)$. Therefore, since the length between each point in the partition is 1, the upper Riemann sum $U(P_n, f) = \sum_{i=1}^{n-1} f(i)$; and similarly $L(P_n, f) = \sum_{i=2}^{n} f(i)$. We use this to show that $\lim_{n\to\infty} \int_1^n f(x)\, dx$ converges by the Cauchy criterion. Given some $\epsilon > 0$, we can find a N such that $\forall p, q \geq N$ the pth and qth partial sums (call them S_p and S_q) of $\sum f(n)$ are within ϵ of each other. Let $M = N + 1$. Then for any $m, n > M$,

$$S_{n-1} = U(P_n, f) \geq \int_1^n f(x)\, dx \geq L(P_n, f) = S_n$$

$$S_{m-1} = U(P_m, f) \geq \int_1^m f(x)\, dx \geq L(P_m, f) = S_m$$

Subtracting these two inequalities, we find

$$S_{n-1} - S_{m-1} \geq \int_1^n f(x)\, dx - \int_1^m f(x)\, dx \geq S_n - S_m$$

We know m, n, $m-1$, and $n-1$ are all greater than N, so $|S_{n-1} - S_{m-1}| \leq \epsilon$ and $|S_n - S_m| \leq \epsilon$. Thus $\left|\int_1^n f(x)\, dx - \int_1^m f(x)\right| dx \leq \epsilon$. This means that $\lim_{n\to\infty} \int_1^n f(x)\, dx = \int_1^\infty f(x)\, dx$ satisfies the Cauchy criterion, so it converges.

Next suppose that $\int_1^\infty f(x)\, dx = \lim_{n\to\infty} \int_1^n f(x)\, dx$ converges. We will show that $\sum_{n=1}^\infty f(n)$ converges. Observe that applying Rudin's theorem 6.12(c) to the integral gives $\lim_{n\to\infty} \int_1^n f(x)\, dx$ $= \lim_{n\to\infty} \sum_{i=1}^{n-1} \int_i^{i+1} f(x)\, dx$. Thus the convergence of $\int_1^\infty f(x)\, dx$ is equivalent to the convergence of the series $\sum_{i=1}^\infty \int_i^{i+1} f(x)\, dx$. Since $f(x)$ is monotonically decreasing, $\int_i^{i+1} f(x)\, dx \geq 1 \inf_{\{i \leq x \leq i+1\}} f(x) = f(i+1)$. Thus the series $\sum_{i=1}^\infty f(i+1)$ converges by the comparison test since it is termwise less than a series that converges. We can certainly add in the term $f(1)$ without affecting the convergence of the series, so $\sum_{i=1}^\infty f(i)$ converges.

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Problem 3: Let f and g be functions such that $f \geq 0$, $g \geq 1$, and $f, g \in \mathscr{R}$ on $[a, b]$. We show that $\frac{f}{g} \in \mathscr{R}$ on $[a, b]$. To do this, we first show that $\frac{1}{g} \in \mathscr{R}$ on $[a, b]$. Since g is Riemann-integrable, it is bounded. We can express this as $1 \leq g \leq M$ for some $M \in \mathbb{R}$. Let $\phi(x) = \frac{1}{x}$. $\phi(x)$ is clearly continuous on the domain $[1, M]$, so we can apply Rudin's theorem 6.11. This tells us that $\phi(g(x)) = \frac{1}{g(x)} \in \mathscr{R}$ on $[a, b]$. We next apply theorem 6.13(a) and find that $f\frac{1}{g} = \frac{f}{g} \in \mathscr{R}$ on $[a, b]$.

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Problem 4: Let f be a bounded function on $[a, b]$ that is continuous except at a countable set E of points. We show that $f \in \mathscr{R}$ on $[a, b]$.

Consider the function $r(x) = \lim_{k\to\infty}(\sup f - \inf f)$ on the interval $(x - \frac{1}{k}, x + \frac{1}{k})$. Define a collection of sets $B_n = \{x \colon r(x) < \frac{1}{n}\}$ for $n \in \mathbb{J}$. Each set B_n is open: consider any $x \in B_n$. Then $r(x) = m$ for some $m < \frac{1}{n}$. That is, $\lim_{k\to\infty}(\sup f - \inf f)$ on $(x - \frac{1}{k}, x + \frac{1}{k})$ is m, which means we can find an interval $(x - \frac{1}{k}, x + \frac{1}{k})$ where $\sup f - \inf f$ is arbitrarily close to m. Since $m < \frac{1}{n}$, we can certainly find such an interval where $(\sup f - \inf f) < \frac{1}{n}$. Every point p in this interval, which is a neighborhood of x, will thus have $r(p) < \frac{1}{n}$, implying $p \in B_n$. Thus every point x of B_n is an interior point, so every B_n is open. Hence, B_n^c is closed. There are a countable number of such sets, so $\bigcup_{n=1}^\infty B_n^c$ is closed.

By definition of continuity, if $r(x) = 0$, $f(x)$ is continuous at x because the sup and inf are equal, and if $r(x) \neq 0$, $f(x)$ cannot be continuous at x. By definition of B_n, B_n^c contains the points x such that $r(x) \geq \frac{1}{n}$, so their union will contain all points where $r(x) > 0$. This is precisely the set E of all points at which f is discontinuous. Since E is countable, each set $B_n^c \subseteq E$ must be finite or countable.

For any m, consider the set B_m^c. This set contains all x for which $r(x) \geq \frac{1}{m}$. It is closed (and bounded, therefore compact), and also countable. We can therefore call its elements $\{x_1, x_2, \dots\}$. For each element x_i, we consider the neighborhood (interval) of radius $\frac{2^{-i}}{m}$ around x_i. These are all open sets, and their union certainly contains B_m^c, which is compact, so there exists a finite subcover that contains B_m^c. This subcover is a finite set of intervals of length $\frac{2^{-i}}{m}$; their total length is strictly less than $\sum_{i=1}^\infty \frac{2^{-i}}{m} = \frac{1}{m}$. We will designate this subcover as the collection of open sets $\{C_1, C_2, \dots, C_k\}$, and let $C = \bigcup_{i=1}^k C_i$. C contains all x for which $r(x) \geq \frac{1}{m}$.

Now we consider the set (call it F) that consists of the interval $[a, b]$ with C removed. Every x in this set has $r(x) < \frac{1}{m}$. By definition of $r(x)$, there exist intervals $(x - \frac{1}{k}, x + \frac{1}{k})$ where $\sup f - \inf f$ is arbitrarily close to $r(x)$; we can thus find such an interval (call it $N(x)$) where $\sup f - \inf f < \frac{1}{m}$ since $r(x) < \frac{1}{m}$. These intervals are open sets, so the union $\bigcup_{x \in F} N(x)$ is an open cover of F. Since C is open and $F = [a, b] \cap C^c$, F is compact. So the open cover can be reduced to a finite subcover $D = N(x_1) \cup N(x_2) \cup \cdots \cup N(x_k)$. Since this still covers F, we can reduce each open interval $N(x_i)$ to a closed subinterval $N'(x_i)$ such that every point in F is in exactly one $N'(x_i)$. The resulting union $D' = \bigcup_{n=1}^{\infty} N'(x_i)$ contains every point in F, and has total length at most $b - a$.

Note that $C \cup D' = [a, b]$. This arrangement of sets C and D' allows us to make the upper and lower Riemann sums arbitrarily close. Letting P be the partition defined by the endpoints of the intervals in C and D', and $M = \sup |f(x)|$, we show:

$$U(P, f) - L(P, f) < \frac{2M}{m} + \frac{b - a}{m}$$

The first term of the inequality comes from the sets in C. The sup and inf on these sets can differ by as much as $2M$, but their length is less than $\frac{1}{m}$. The second term comes from the sets in D'. The sup and inf on these sets can only differ by less than $\frac{1}{m}$ (since they only contain points where $r(x) < \frac{1}{m}$), and the total length of these sets is at most $b - a$. Since m was arbitrary, given any $\epsilon > 0$, we can choose $m = \left\lceil \frac{2M+b-a}{\epsilon} \right\rceil$, which makes the upper and lower Riemann sums within ϵ of each other. Thus, $f \in \mathscr{R}$ on $[a, b]$.

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