18.100b **Dan Ports**

Problem Set 8

50/50

2003/05/02 drkp@mit.edu

Problem 1:

Part a Note that since $\frac{1}{p} + \frac{1}{q} = 1$ and p and q are both positive, both p and q must be greater than 1 (else their reciprocals could not add to 1). Let $x = \frac{u^p}{v^q}$ (we assume both u and v are non-zero so x is defined). Assume without loss of generality that $u^p > v^q$ (by the symmetry of the problem, we can exchange u^p and v^q; we will consider the $u^p = v^q$ case later). Then $x > 1$. Consider the function $f(x) = \frac{x}{p} - x^{\frac{1}{p}}$. Its derivative is $f'(x) = \frac{1}{p} - \frac{1}{p}x^{\frac{1}{p}-1} = \frac{1}{p} - \frac{1}{p}x^{-q} = \frac{1}{p}(1 - x^{-q})$. Certainly $\frac{1}{p} > 0$. Since $q > 0$ and $x > 1$, $x^{-q} < 1$ and thus $(1 - x^{-q}) > 0$. So $f'(x) > 0$ and $f(x)$ is strictly increasing on $(1, \infty)$. Note that $f(1) = \frac{1}{p} - 1 = -\frac{1}{q}$. Thus on $(1, \infty)$, $f(x) > -\frac{1}{q}$, so $f(x) + \frac{1}{q} > 0$:

$$
\begin{aligned}
\frac{x}{p} - x^{\frac{1}{p}} + \frac{1}{q} &> 0 \\
\frac{1}{p}\frac{u^p}{v^q} - \frac{u}{v^{\frac{q}{p}}} + \frac{1}{q} &> 0 \\
\frac{1}{p}u^p - uv^{-\frac{q}{p}+q} + \frac{1}{q}v^q &> 0 \\
\frac{u^p}{p} + \frac{v^q}{q} &> uv^{-\frac{q}{p}+q} = uv^{-(q-1)+q} = uv
\end{aligned}
$$

We need also to consider the cases where x is undefined. If $u \neq v = 0$, then the inequality is simply $0 < \frac{u^p}{p}$, which is clearly true, and similarly if $v \neq u = 0$. If $u = v = 0$, then both sides of the inequality are zero and equality holds.

Next we consider the possibility that $u^p = v^q$. In this case, $x = 1$. Dividing the inequality by v^q and substituting x as before gives $\frac{x}{p} + \frac{1}{q} \geq x^{\frac{1}{p}}$. With $x = 1$, the inequality becomes an equality: $\frac{1}{p} + (1 - \frac{1}{p}) = 1$. Thus equality holds if $x = 1$, in which case $u^p = v^q$. If $x \neq 1$, we showed above that the strict inequality $\frac{u^p}{p} + \frac{v^q}{q} > uv$ holds. Thus, equality holds if and only if $u^p = v^q$.

Part b Since $f, g \in \mathscr{R}$, $fg \in \mathscr{R}$. From part a, $\forall x \in [a, b]$ $fg \leq \frac{f^p}{p} + \frac{g^q}{q}$. Thus, applying theorem 6.12(b),

$$\int_a^b fg\,dx \leq \frac{1}{p}\int_a^b f^p\,dx + \frac{1}{q}\int_a^b g^q\,dx = \frac{1}{p} + \frac{1}{q} = 1$$

Part c If f and g are complex-valued Riemann-integrable functions, the functions $|f|$ and $|g|$ are positive real-valued Riemann-integrable functions, and by theorem 6.13(b), $\left|\int_a^b fg\,dx\right| \leq \int_a^b |fg|\,dx \leq \int_a^b |f||g|\,dx$. Let $\alpha = (\int_a^b |f|^p\,dx)^{\frac{1}{p}}$ and $\beta = (\int_a^b |g|^q\,dx)^{\frac{1}{q}}$. Suppose first that α and β are both non-zero. Then let $h = \frac{|f|}{\alpha}$ and $j = \frac{|g|}{\beta}$. Then $\int_a^b h^p\,dx = \frac{1}{\int_a^b |f|^p\,dx}\int_a^b |f|^p\,dx = 1$, and by the same

10/10

reasoning $\int_a^b j^q\,dx = 1$. By part b:

$$\begin{aligned}
\int_a^b hj\,dx = \int_a^b \frac{|f|}{\alpha}\frac{|g|}{\beta}\,dx &\leq 1 \\
\int_a^b |f||g|\,dx &\leq \alpha\beta \\
\int_a^b |f||g|\,dx &\leq \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g|^q\,dx\right)^{\frac{1}{q}} \\
\left|\int_a^b fg\,dx\right| &\leq \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g|^q\,dx\right)^{\frac{1}{q}}
\end{aligned}$$

We also need to consider the case where either α or β is zero. Suppose $\alpha = 0$. Then $\int_a^b |f|^p\,dx = 0$. So $\forall x\ |f(x)|^p = 0$ because $|f(x)|^p$ is everywhere non-negative and its integral is zero (this is a result from the last problem set). Thus $|f| = 0$, so $\left|\int_a^b fg\,dx\right| \leq \int_a^b |f||g|\,dx = 0$. This means $\left|\int_a^b fg\,dx\right|$ is zero (since it certainly cannot be negative). Since $\alpha = (\int_a^b |f|^p\,dx)^{\frac{1}{p}}$ is also zero, the inequality reduces to $0 \leq 0$, which is obviously true. The same argument can be applied if $\beta = 0$. So the desired inequality $\left|\int_a^b fg\,dx\right| \leq \left(\int_a^b |f|^p\,dx\right)^{\frac{1}{p}}\left(\int_a^b |g|^q\,dx\right)^{\frac{1}{q}}$ always holds.

Problem 2:

Part a The $n = 0$ case reduces to $f(x) = f(a) + \int_a^x f'(x)\,dx$. This follows from the fundamental theorem of calculus, which tells us $\int_a^x f'(x)\,dx = f(x) - f(a)$. Applying this,

$$f(x) = f(a) + f(x) - f(a) = f(a) + \int_a^x f'(x)\,dx$$

.

Part b Assume by induction that $f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_n(x)$. Let $g(t) = -\frac{(x-t)^n}{n}$ and $h(t) = f^{(n)}(t)$. Then $g'(t) = (x-t)^{n-1}$ and $h'(t) = f^{(n+1)}(t)$. Then we can write $R_n(x) = \frac{1}{(n-1)!}\int_a^x g'(t)h(t)\,dt$. Applying integration by parts,

$$\begin{aligned}
R_n(x) &= \frac{1}{(n-1)!}\left[g(x)h(x) - g(a)h(a) - \int_a^x g(t)h'(t)\,dt\right] \\
&= \frac{1}{(n-1)!}\left[-\frac{(x-x)^n}{n}f^{(n)}(x) - \left(-\frac{(x-a)^n}{n}f^{(n)}(a)\right) - \int_a^x \left(-\frac{(x-t)^n}{n}f^{(n+1)}(t)\right)dt\right] \\
&= \frac{1}{(n-1)!}\left[\frac{(x-a)^n}{n}f^{(n)}(a) + \int_a^x \frac{(x-t)^n}{n}f^{(n+1)}(t)\,dt\right] \\
&= \frac{1}{n!}(x-a)^n f^{(n)}(a) + R_{n+1}(x)
\end{aligned}$$

So we have shown that

$$\begin{aligned}
f(x) &= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{1}{n!}\frac{(x-a)^n}{n}f^{(n)}(a) + R_{n+1}(x) \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n+1}(x)
\end{aligned}$$

10/10

Thus we have proven the inductive hypothesis for n, and by induction the theorem holds for all $n \in \mathbb{N}$.

Problem 3: Let $\{f_n\}$ and $\{g_n\}$ be sequences of bounded functions that converge uniformly on E. We will show that their product $\{f_n g_n\}$ converges via the Cauchy criterion. The sequences are bounded, so we can find some M_f such that $\forall n \in \mathbb{J}\ \forall x \in E\ |f_n(x)| < M_f$, and similarly M_g such that $\forall n \in \mathbb{J}\ \forall x \in E\ |g_n(x)| < M_g$; let $M = \max\{M_f, M_g\}$. Let $\epsilon > 0$ be given. Then by the uniform convergence of $\{f_n\}$, we can find a N_f such that $\forall n, m \geq N_f\ \forall x \in E\ |f_n(x) - f_m(x)| \leq \frac{\epsilon}{2M}$, and a N_g similarly chosen by the same criteria on g. Let $N = \max\{N_f, N_g\}$. Then for any $n, m \geq N$ and any $x \in E$, consider $|f_n(x)g_n(x) - f_m(x)g_m(x)|$. By the triangle inequality, this is less than or equal to $|f_n(x)g_n(x) - f_n(x)g_m(x)| + |f_n(x)g_m(x) - f_m(x)g_m(x)| = |f_n(x)||g_n(x) - g_m(x)| + |f_n(x) - f_m(x)||g_m(x)|$. But $|f_n(x)|$ and $|g_m(x)|$ are both less than M by their boundedness, and $|f_n(x) - f_m(x)|$ and $|g_n(x) - g_m(x)|$ are less than or equal to $\frac{\epsilon}{2M}$ by our choice of N. So this quantity is less than $M\frac{\epsilon}{2M} + M\frac{\epsilon}{2M} = \epsilon$. Thus the sequence $\{f_n g_n\}$ satisfies the Cauchy criterion for uniform convergence.

We now show that this is not true in the unbounded case. Consider two sequences of functions defined on the interval $[0, \infty)$. Define $f_n(x) = x^2$ for all n. Clearly, $\{f_n\}$ is uniformly convergent to $f(x) = x^2$ (since it does not even depend on n). Define $g_n(x)$ to be zero for $x \in [0, n]$ and $\frac{1}{x}$ for $x \in (n, \infty)$. Then $\{g_n\}$ converges uniformly to $g(x) = 0$: given any $\epsilon > 0$, we let $N = \lceil \frac{1}{\epsilon} \rceil$. Then for any $n \geq N$, $|g_n(x)|$ is everywhere less than ϵ because g_n is defined to be zero for $x \leq n$, and for $x > n$ $g_n(x) = \frac{1}{x} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$. Next consider the product of the sequences:

$$f_n g_n = \begin{cases} 0 & 0 \leq x \leq n \\ x^2 \frac{1}{x} = x & x > n \end{cases}$$

10/10

The sequence $\{f_n g_n\}$ certainly converges to $f(x)g(x) = 0$, but this convergence is not uniform. Given any $\epsilon > 0$ and any $n \in \mathbb{J}$, we will find a x such that $f_n(x)g_n(x) - f(x)g(x) > \epsilon$. Specifically, consider $x = \max\{\epsilon + 1, n + 1\}$. Then $f_n(x)g_n(x) = x > \epsilon$. So $f_n(x)g_n(x) - f(x)g(x) = x - 0 > \epsilon$. Thus $\{f_n g_n\}$ cannot satisfy the definition of uniform convergence.

Problem 4: We consider the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2 x}$, with $x \geq 0$. Fix any $x > 0$, and note that the series is termwise less than $\sum_{n=1}^{\infty} \frac{1}{n^2 x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n^2}$. Thus by the comparison test, since $\sum \frac{1}{n^2}$ converges, the series must converge for any $x > 0$. Since we are restricted to considering non-negative x, every term of the series is always positive, so the series converges absolutely for $x > 0$. The series obviously diverges for $x = 0$.

We next show that the series converges uniformly on any interval of the form $[x, \infty)$ (for $x > 0$). Write $f_n(x) = \frac{1}{1+n^2 x}$; then $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Let $x > 0$ be given, and consider the set $E = [x, \infty)$. For any $t \in E$, $t \geq x$, and $|f_n(t)| = \left|\frac{1}{1+n^2 t}\right| \leq \left|\frac{1}{1+n^2 x}\right| = |f_n(x)|$ By the previous result, the series $\sum_{n=1}^{\infty} f_n(x)$ converges. Therefore, by the Weierstrass m-test, $\sum_{n=1}^{\infty} f_n(t) = f(t)$ converges uniformly on E. So for any $x > 0$, f converges uniformly on $[x, \infty)$.

The series fails to converge uniformly on any interval $(0, x]$ (with $x > 0$). Let $x > 0$ be given and consider $E = (0, x]$. We will show that $\sum f_n(t)$ fails to satisfy the Cauchy criterion for uniform convergence on E. Let $p, q \in \mathbb{J}$ be given ($p \neq q$), and assume without loss of generality that $p < q$. Then consider $|\sum_{n=1}^{p} f_n(t) - \sum_{n=1}^{q} f_n(t)| = \sum_{n=p}^{q} f_n(t)$. The first term of this summation is $f_p(t) = \frac{1}{1+p^2 t}$. Since we are considering $t \in (0, x]$, we can take $u = \min\{x, \frac{1}{p^2}\}$ and we will have a u that satisfies both $u \in E$ and $u \le \frac{1}{p^2}$. Then $f_p(u) \ge \frac{1}{1+p^2\frac{1}{p^2}} = \frac{1}{2}$. So $\sum_{n=p}^{q} f_n(u)$ is certainly bounded below by its first term, $f_p(u) = \frac{1}{2}$. So we have shown that for any p, q we can find a $u \in E$ such that $|\sum_{n=1}^{p} f_n(y) - \sum_{n=1}^{q} f_n(y)| > \frac{1}{2}$. Thus $\sum f_n(t)$ fails to satisfy the Cauchy criterion for uniform convergence, and therefore f does not converge uniformly on $(0, x]$ for any $x > 0$.

Next we show f is continuous on the domain $(0, \infty)$ where it converges. Consider any $x > 0$. We will show that f is continuous at x. Note that by the density of the real numbers, we can find some a satisfying $0 < a < x$. Then $x \in (a, \infty)$. From above, f is uniformly convergent on (a, ∞). Also, for each n, $f_n(x) = \frac{1}{1+n^2x}$ is clearly continuous on (a, ∞). Thus, since $f = \sum f_n$ is a uniformly convergent sum of continuous functions on (a, ∞), it is continuous on (a, ∞). Thus it is continuous at x. Since x was arbitrary, f is continuous on $(0, \infty)$.

Finally, we show that f is unbounded. For any $M > 0$, we will find a point $x \in (0, \infty)$ such that $f(x) > M$. Let $k = \lceil 2M \rceil$. Then choose $x = \frac{1}{k^2}$. Thus for $n \le k$, $f_n(x) = \frac{1}{1+n^2\frac{1}{k^2}} \ge \frac{1}{2}$. So $f(x) = \sum_{n=1}^{\infty} f_n(x) > \sum_{n=1}^{k} f_n(x) \ge \frac{1}{2}k \ge M$. Thus f is unbounded.

10/10

Problem 5: We first show that if $\{f_n\} = \{g_n + h_n\}$ and $\{g_n\}$ and $\{h_n\}$ are uniformly convergent sequences of functions, then $\{f_n\}$ is also uniformly convergent. Let g and h be the limit functions of $\{g_n\}$ and $\{h_n\}$ respectively. We will show that $\{f_n\}$ converges uniformly to $g + h$. Let $\epsilon > 0$ be given. By the uniform convergence of $\{g_n\}$ we can find a N_g such that $\forall n \ge N_g$ $\forall x$ $|g_n(x)| \le \frac{\epsilon}{2}$. We can find a N_h defined equivalently for h_n. Then let $N = \max\{N_g, N_h\}$ and $\forall n \ge N$, consider $|f_n(x) - (g(x) + h(x))| = |g_n(x) + h_n(x) - g(x) - h(x)| \le |g_n(x) - g(x)| + |h_n(x) - h(x)| \le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. So $\{f_n\}$ converges uniformly to $g + h$. By considering the sequences of partial sums, we easily see that this theorem applies to series as well: if $\sum g_n$ and $\sum h_n$ converge uniformly, and $f_n = g_n + h_n$ then $\sum f_n$ converges uniformly.

We use this result to show that $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval. Observe that this series can be written as the sum of two series: $\sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2} + \sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2}$. To show that it converges uniformly in every bounded interval, we need to show that both of these two series converge uniformly. The first series is $\sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$. Since we are considering a bounded interval, $\exists M \in \mathbb{R}$ such that $\forall x$ $|x| \le M$. Thus $\left|(-1)^n \frac{x^2}{n^2}\right| \le \frac{M^2}{n^2}$. Since M is constant, $\sum \frac{M^2}{n^2}$ converges, so by the Weierstrass m-test $\sum_{n=1}^{\infty} (-1)^n \frac{x^2}{n^2}$ converges uniformly on any bounded interval.

The second series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$, which is independent of x. As a series in n, it converges (it is the alternating harmonic series). Since it is independent of x, it certainly converges uniformly on any interval. By our result above, since $\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ is the sum of two series that converge uniformly on any bounded interval, it converges uniformly on any bounded interval.

$\sum_{n=1}^{\infty} (-1)^n \frac{x^2+n}{n^2}$ does not converge absolutely for any value of x. Observe that $\left|(-1)^n \frac{x^2+n}{n^2}\right| = \frac{x^2+n}{n^2} = \frac{\frac{x^2}{n}+1}{n}$. Thus for any x the resulting series is termwise greater than or equal to $\frac{1}{n}$, which diverges, so by the comparison test it diverges. ✓

10/10