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	Problem Set 1	
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Problem 1:

Analytic proof Using the definition of binomial coefficients, we expand $\binom{2n}{n}$:

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \prod_{i=0}^{n-1} \frac{2n-i}{n-i}$$

Next, observe that (for $0 \le i \le n-1$), $\frac{2n-i}{n-i} \ge 2$. The $i = 0$ case reduces to $\frac{2n}{n} = 2$, and the terms of the product are monotonically increasing:

$$\frac{2n-(i+1)}{n-(i+1)} - \frac{2n-i}{n-i} = \frac{(2n-i-1)(n-i)-(2n-i)(n-i-1)}{(n-i-1)(n-i)} = \frac{n}{(n-i-1)(n-i)} > 0$$

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Therefore each term is at least 2. There are n terms, each at least 2, so we have shown that $\binom{2n}{n} \ge 2^n$.

Combinatorial proof We will prove the inequality by identifying a set with 2^n elements that is a subset of a set with $\binom{2n}{n}$ elements. Consider the set of possible n-digit strings made up of the characters A and B. There are 2^n such strings. Now consider the problem of choosing n elements from the set $\{A_1, A_2, \ldots, A_n, B_1, B_2, \ldots, B_n\}$. Since there are $2n$ total elements, there are $\binom{2n}{n}$ possible choices. Each n-digit string from the first set can be represented uniquely as a set of elements from the second set: for each character in the string, the element corresponding to that character and that position in the string is chosen (e.g. the string $ABAA$ becomes the set A_1, B_2, A_3, A_4). Thus the set of n-strings (2^n total) is contained within the set of n-combinations of the $2n$ set ($\binom{2n}{n}$ total). So we have proven $2^n \le \binom{2n}{n}$. 10

Problem 2:

We will use a combinatorial argument to show that $\sum_{n_1+n_2+n_3=n} \binom{n}{n_1,n_2,n_3} = 3^n$. First, note that for any such n_1, n_2, n_3, $\binom{n}{n_1,n_2,n_3}$ is the number of ways to partition n elements into three distinct sets of size n_1, n_2, and n_3. Next, consider the problem of partitioning n elements into three distinct sets of any size (including size zero). The possible set sizes are the possible values of n_1, n_2, and n_3 that add up to n, and so the total number of partitions is the sum over all these values: $\sum_{n_1+n_2+n_3=n} \binom{n}{n_1,n_2,n_3}$. But we can also compute the total number of such partitions by assigning a value between 1 and 3 to each of the n elements, corresponding to the set they are placed in. This creates 3^n partitions. Thus we have shown that $\sum_{n_1+n_2+n_3=n} \binom{n}{n_1,n_2,n_3} = 3^n$. 20

Problem 3:

Part a Choosing 4 cards from 52, there are $\binom{52}{4} = \frac{52!}{4!(52-4)!} = 270725$ possible hands.

Part b There are 13 cards in each suit, so the total number of hands (ignoring ordering) with one card from each suit is $13^4 = 28561$. 20

Problem 4:

First, consider a binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Define the function $p_2(x)$ to be the number of times x is divisible by 2. We know that $p_2(n!) \geq p_2(k!) + p_2((n-k)!)$, since the binomial coefficients are integers. When $p_2(n!) > p_2(k!) + p_2((n-k)!)$, $\binom{n}{k}$ is divisible by 2. Otherwise, when $p_2(n!) = p_2(k!) + p_2((n-k)!)$, all factors of 2 cancel and $\binom{n}{k}$ is odd.

A printout of some testing in Matlab is attached. This script makes use of the recurrence

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

(Ross, p.8), along with the base cases $\binom{0}{0} = 1$, $\binom{n}{0} = 1$, and $\binom{n}{n} = 1$. All computations are performed modulo 2. The result is a Sierpinski triangle. Computed to the first hundred rows, we find 3799 even numbers and 1225 odds: 75.62% of the numbers are even. So we conclude that most binomial coefficents are even.

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```
>> odds=0

odds =

     0

>> evens=0

evens =

     0

>> A = zeros(100);

>>  for i=(1:100)
for j=(1:i)
if (i==1 || j==1 || j==i)
A(i,j)=1;
else
A(i,j)=mod(A(i-1,j-1)+A(i-1,j),2);
end
if (A(i,j)==1)
odds = odds+1;
else
evens = evens+1;
end
end
end

>> odds

odds =

        1255

>> evens

evens =

        3799


A =

  Columns 1 through 13

     1     0     0     0     0     0     0     0     0     0     0     0     0
     1     1     0     0     0     0     0     0     0     0     0     0     0
     1     0     1     0     0     0     0     0     0     0     0     0     0
     1     1     1     1     0     0     0     0     0     0     0     0     0
     1     0     0     0     1     0     0     0     0     0     0     0     0
     1     1     0     0     1     1     0     0     0     0     0     0     0
     1     0     1     0     1     0     1     0     0     0     0     0     0
     1     1     1     1     1     1     1     1     0     0     0     0     0
     1     0     0     0     0     0     0     0     1     0     0     0     0
     1     1     0     0     0     0     0     0     1     1     0     0     0
     1     0     1     0     0     0     0     0     1     0     1     0     0
     1     1     1     1     0     0     0     0     1     1     1     1     0
     1     0     0     0     1     0     0     0     1     0     0     0     1
     1     1     0     0     1     1     0     0     1     1     0     0     1
     1     0     1     0     1     0     1     0     1     0     1     0     1
     1     1     1     1     1     1     1     1     1     1     1     1     1
     1     0     0     0     0     0     0     0     0     0     0     0     0

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