50
20
29
99

18.440 **Dan Ports**

Problem Set 6

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Problem 1:

Part a The boundaries of the probability space are the lines $y = 0$, $x = 1$, and $y = x$. The lines dividing the regions are $y = y - x$ (or $y = \frac{x}{2}$), $y = 1 - x$, and $y - x = 1 - x$, or ($y = 2x - 1$). See the attached sketch.

Part b

$$X = \begin{cases} y & R_{ABC} \\ y & R_{ACB} \\ y - x & R_{BAC} \\ y - x & R_{BCA} \\ 1 - x & R_{CAB} \\ 1 - x & R_{CBA} \end{cases}$$

your notations are backwards

Part c

$$\begin{aligned}
\Pr[R_{ABC}] &= 2\int_{\frac{2}{3}}^{1}\int_{\frac{x}{2}}^{2x-1} dy\,dx = \frac{1}{6} \\
\Pr[R_{ACB}] &= 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{1-x}^{x} dy\,dx + 2\int_{\frac{2}{3}}^{1}\int_{2x-1}^{x} dy\,dx = \frac{1}{18} + \frac{1}{9} = \frac{3}{18} \quad = \frac{1}{6} \\
\Pr[R_{BAC}] &= 2\int_{\frac{2}{3}}^{1}\int_{1-x}^{\frac{x}{2}} dy\,dx = \frac{1}{6} \\
\Pr[R_{BCA}] &= 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{0}^{2x-1} dy\,dx + 2\int_{\frac{2}{3}}^{1}\int_{1-x}^{0} dy\,dx = \frac{1}{18} + \frac{1}{9} = \frac{3}{18} \quad = \frac{1}{6} \\
\Pr[R_{CAB}] &= 2\int_{0}^{\frac{1}{2}}\int_{\frac{x}{2}}^{x} dy\,dx + 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{\frac{x}{2}}^{1-x} dy\,dx = \frac{1}{8} + \frac{1}{24} = \frac{1}{6} \\
\Pr[R_{CAB}] &= 2\int_{0}^{\frac{1}{2}}\int_{\frac{x}{2}}^{x} dy\,dx + 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{\frac{x}{2}}^{1-x} dy\,dx = \frac{1}{8} + \frac{1}{24} = \frac{1}{6} \\
\Pr[R_{CBA}] &= 2\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{x}{2}} dy\,dx + 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{2x-1}^{\frac{x}{2}} dy\,dx = \frac{1}{8} + \frac{1}{24} = \frac{1}{6}
\end{aligned}$$

√ **Part d**

$$
\begin{aligned}
\mathrm{E}[X] &= \int\!\!\int_S X\,dA \\
&= \int\!\!\int_{R_{ABC}} y\,dA + \int\!\!\int_{R_{ACB}} y\,dA + \int\!\!\int_{R_{BAC}} x-y\,dA \\
&\quad + \int\!\!\int_{R_{BCA}} x-y\,dA + \int\!\!\int_{R_{CAB}} 1-x\,dA + \int\!\!\int_{R_{CBA}} 1-x\,dA \\
&= 2\int_{\frac{2}{3}}^{1}\int_{\frac{x}{2}}^{2x-1} y\,dy\,dx \\
&\quad +2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{1-x}^{x} y\,dy\,dx + 2\int_{\frac{2}{3}}^{1}\int_{2x-1}^{x} y\,dy\,dx \\
&\quad +2\int_{\frac{2}{3}}^{1}\int_{1-x}^{\frac{x}{2}} x-y\,dy\,dx \\
&\quad +2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{0}^{2x-1} x-y\,dy\,dx + 2\int_{\frac{2}{3}}^{1}\int_{1-x}^{0} x-y\,dy\,dx \\
&\quad +2\int_{0}^{\frac{1}{2}}\int_{\frac{x}{2}}^{x} 1-x\,dy\,dx + 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{\frac{x}{2}}^{1-x} 1-x\,dy\,dx \\
&\quad +2\int_{0}^{\frac{1}{2}}\int_{0}^{\frac{x}{2}} 1-x\,dy\,dx + 2\int_{\frac{1}{2}}^{\frac{2}{3}}\int_{2x-1}^{\frac{x}{2}} 1-x\,dy\,dx \\
&= \frac{11}{108}+\frac{1}{36}+\frac{2}{27}+\frac{11}{108}+\frac{1}{36}+\frac{2}{27}+\frac{1}{12}+\frac{1}{54}+\frac{1}{12}+\frac{1}{54} \\
&= \frac{11}{18}
\end{aligned}
$$

50

Problems 2 & 3: We consider the general case of service stations located at points a, b, c ($0 \le a \le b \le c \le 100$). Then the expected distance is

$$
\begin{aligned}
100 \cdot \mathrm{E}[X] &= \int_0^a a-x\,dx + \int_a^{a+\frac{b-a}{2}} x-a\,dx \\
&\quad + \int_{a+\frac{b-a}{2}}^{b} b-x\,dx + \int_b^{b+\frac{c-b}{2}} x-b\,dx \\
&\quad + \int_{b+\frac{c-b}{2}}^{c} c-x\,dx + \int_c^{100} x-c\,dx \\
&= \frac{b^2}{2} + b\left(-\frac{c}{2}-\frac{a}{2}\right) + \frac{3c^2}{4} - \frac{100c}{2} + \frac{3a^2}{4} + \overset{2500}{5000}
\end{aligned}
$$

√ o.k.

|| o.k.

For the case in which the service stations are located in towns A and B and exactly in between, $a = 0$, $b = 50$, and $c = 100$. Using the formula above, $\mathrm{E}[X] = 25$.

When the service stations are located at $a = 25$, $b = 50$, $c = 75$, $\mathrm{E}[X] = 17.375$. So this is a better location than before.

To find the optimum locations, we take the partial derivative with respect to a and set it equal to zero:

$$\frac{3a}{2} - \frac{b}{2} = 0$$

So $a = \frac{b}{3}$. Next, taking the partial derivative with respect to b and setting it equal to zero:

$$\frac{5b}{6} - \frac{c}{2} = 0$$

So $b = \frac{3c}{5}$. Finally, taking the partial derivative with respect to c and setting it equal to zero:

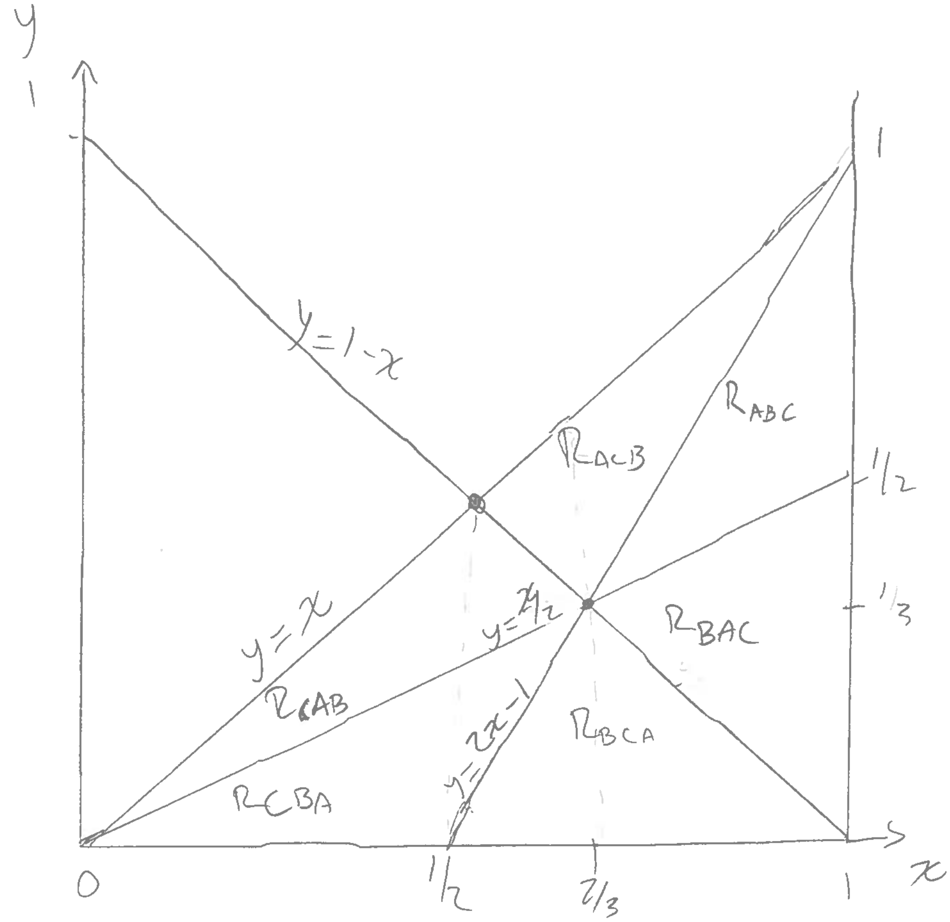
$$\frac{6c}{5} - 100 = 0$$

So we conclude $c = \frac{5}{6}100 \approx 83.33$. Thus $b = 50$ and $a \approx 16.67$. This is the optimum placement of stations.

for problem 2: 20 pts
problem 3: 29 pts

D Ports

DS6 - 1


y
1
y = 1 - x
1
R_ACB
R_ABC
1/2
1/3
y = x
y = x/2
R_BAC
R_CAB
y = 2x - 1
R_BCA
R_CBA
0
1/2
2/3
1
x