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18.440 **Dan Ports**

Problem Set 7

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Problem 1:

Part a

$$\begin{aligned} \mathrm{E}[R] &= \int\int_S R(x,y)\,dA = \int_0^{2\pi}\int_0^1 \frac{r^2}{\pi}\,dr\,d\theta \\ &= \int_0^{2\pi} \frac{1}{3\pi}\,d\theta = \frac{2}{3} \end{aligned}$$

√

Part b The cumulative distribution function $F_R(r)$ is the area the subcircle with radius r divided by π:

$$F_R(r) = \frac{\pi r^2}{\pi} = r^2$$

and so √

$$f_R(r) = 2r$$

Part c

$$\mathrm{E}\left[R^2\right] = \int_0^1 r^2 f_R(r)\,dr = \int_0^1 2r^3\,dr = \frac{2}{4}1^4 = \frac{1}{2}$$

√

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Part d We expect the expected value of R to be more than $\frac{1}{2}$. The area of the subcircle of radius $\frac{1}{2}$ is $\frac{1}{4}$th the total area of the circle. So there are more points in the sample space whose distance from the center is greater than $\frac{1}{2}$ than there are whose distance from the center is less than $\frac{1}{2}$. √

Problem 2:

Part a Since $f_X(x)$ is a probability density function, $\int_a^\infty f_X(x)\,dx = \lim_{b\to\infty}\int_a^b f_X(x)\,dx = 1$. So

$$\lim_{b\to\infty} b(1 - F_X(b)) = \lim_{b\to\infty} b - b\int_a^b f_X(x)\,dx = 0$$

– no

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Part b Applying integration by parts, for any $b \geq a$,

$$\begin{aligned} \int_a^b x f_X(x)\,dx &= bF_X(b) - aF_X(a) - \int_a^b F_X(x)\,dx \\ &= -\int_a^b F_X(x)\,dx + b - a - b(1 - F_X(b)) + a(1 - F_X(a)) \\ &= \int_a^b 1 - F_X(x)\,dx - b(1 - F_X(b)) + a(1 - F_X(a)) \end{aligned}$$

√

Part c

$$\begin{aligned}
\mathrm{E}[X] &= \int_a^\infty x f_X(x)\,dx \\
&= \lim_{b\to\infty}\int_a^b x f_X(x)\,dx \\
&= \lim_{b\to\infty}\int_a^b 1 - F_X(x)\,dx - b(1-F_X(b)) + a(1-F_X(a)) \\
&= \lim_{b\to\infty}\int_a^b 1 - F_X(x)\,dx + a
\end{aligned}$$

The last step is true because $\lim_{b\to\infty} b(1-F_X(b)) = 0$ from part a, and $F_X(a) = 0$ because it is the cumulative distribution function of a random variable that always takes values greater than or equal to a. So we can write

$$\mathrm{E}[X] = \int_a^\infty 1 - F_X(x)\,dx + a$$

The improper integral converges because it is the limit $\lim_{b\to\infty}\int_a^b 1 - F_X(x)\,dx$ above, which exists because it is equal to $\mathrm{E}[X] - a$, and $\mathrm{E}[X]$ exists.

√

Part d Taking $a' < a$, we obtain

$$\begin{aligned}
\mathrm{E}[X] &= \int_{a'}^\infty 1 - F_X(x)\,dx + a' \\
&= \int_{a'}^\infty 1 - F_X(x)\,dx + \int_a^\infty 1 - F_X(x)\,dx + a' \\
&= \int_{a'}^\infty 1 - \int_{a'}^\infty F_X(x)\,dx + \int_a^\infty 1 - F_X(x)\,dx + a' \\
&= a - a' + \int_a^\infty 1 - F_X(x)\,dx + a' \\
&= \int_a^\infty 1 - F_X(x)\,dx + a
\end{aligned}$$

√ 30

since $F_X(x) = 0$ for $x < a$ because X takes only values greater or equal to a ~~**Problem 3:**~~

Problem 3:

When tossing a fair coin 900 times, the expected number of heads is $900\frac{1}{2} = 450$ and the variance is $900\frac{1}{2}(1-\frac{1}{2}) = 225$. We approximate the experiment with a normal random variable with the same mean and variance. Since the standard deviation is 15, the probability that the number of heads will be within $10 = \frac{2}{3}\sigma$ of the mean is

$$\Phi(\frac{2}{3}) - (1 - \Phi(\frac{2}{3})) \approx .4972$$

√ 10

Problem 4:

Let Y be the random variable with probability density function $f_Y(x) = \frac{1}{3}(e^{-x} + e^{-\frac{1}{2}x})$ for $x > 0$ and 0 otherwise. Note that $\int_0^\infty f_Y(x)\,dx = \frac{1}{3}\int_0^\infty e^{-x} + e^{-\frac{1}{2}x}\,dx = \frac{1}{3}3 = 1$, so this is a valid probability density function.

The cumulative distribution function is $F_Y(x) = \int_0^x f_Y(t)\,dt = 1 - \frac{2}{3}e^{-\frac{x}{2}} - \frac{1}{3}e^{-x}$, so the hazard rate function is

$$\lambda_Y(x) = \frac{f_Y(x)}{1 - F_Y(x)} = \frac{\frac{1}{3}\left(e^{-x} + e^{-\frac{x}{2}}\right)}{\frac{2}{3}e^{-\frac{x}{2}} + \frac{1}{3}e^{-x}} = \frac{e^{\frac{x}{2}} + 1}{2e^{\frac{x}{2}} + 1}$$

The derivative of the hazard rate is

$$\lambda_Y'(x) = \frac{-e^{\frac{x}{2}}}{2\left(2e^{\frac{x}{2}} + 1\right)^2}$$

which is clearly negative for all $x > 0$. So the hazard rate function is everywhere decreasing and the random variable Y thus has the desired property that $\forall s, t > 0 \ \Pr\left[Y \geq s + t | Y \geq t\right] > \Pr\left[\underset{Y}{\cancel{X}} > s\right]$.

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