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	Problem Set 8	
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Problem 1:

Part a The cumulative distribution function

$$\begin{aligned} F_{XY}(a,b) &= \iint \frac{1}{\pi}\{S \mid x \le a, y \le b\} \\ &= \int_{-1}^{\min\{a,1\}} \int_{-\sqrt{1-a^2}}^{\min\{b,\sqrt{1-a^2}\}} \frac{1}{\pi}\, dy\, dx \end{aligned}$$

assuming $a > -1$ and $b > -1$.

By cases:

If $a \le -1$ or $b \le -1$, $F_{XY}(a,b) = 0$. If $-1 < a \le 1$ and $-1 < b \le \sqrt{1-a^2}$,

$$F_{XY}(a,b) = \int_{-1}^{a} \int_{-\sqrt{1-x^2}}^{b} \frac{1}{\pi}\, dy\, dx = \frac{2\sin^{-1} a + 2a\sqrt{1-a^2} + 4ab + 4b + \pi}{4\pi}$$

If $-1 < a \le 1$ and $\sqrt{1-a^2} < b \le 1$,

$$\begin{aligned} F_{XY}(a,b) &= \int_{-1}^{a} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy\, dx \\ &= \frac{2\sin^{-1} a + 2a\sqrt{1-a^2} + \pi}{2\pi} \end{aligned}$$

If $a > 1$ and $-1 \le b < 1$

$$F_{XY}(a,b) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{b} \frac{1}{\pi}\, dy\, dx = \frac{4b + \pi}{2\pi}$$

(-5)

If $a > 1$ and $b > 1$, $F_{XY}(a,b) = 1$.

Part b Because the distribution is uniform over a disc of radius 1, $f_{XY}(a,b) = \frac{1}{\pi}$ if $a^2 + b^2 < 1$ and zero otherwise. ✓

Part c The expected value is

$$\begin{aligned} \mathrm{E}[|X|] &= \iint_S |X|\, dA \\ &= \frac{1}{\pi} \int_0^{2\pi} \int_0^1 |r\cos\theta| r\, dr\, d\theta \\ &= \frac{4}{3\pi} \approx .4244 \end{aligned}$$

✓

Part d We expect $\mathrm{E}[|X|]$ to be less than $\frac{1}{2}$ because the area with $|X| < \frac{1}{2}$ is larger than the area with $|X| > \frac{1}{2}$; they both have width 1, but the inner section has greater height. ✓

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Problem 2:

Part a If both i and j are non-zero, $p_{XY}(i,j)=0$ since this would require both girls and boys to be at the post office, and the bus only picks up one or the other.

For $i>0$,

$$p_{XY}(i,0)=pe^{-\lambda_B}\frac{\lambda_B^i}{i!}$$

and similarly for $j>0$

$$p_{XY}(0,j)=(1-p)e^{-\lambda_G}\frac{\lambda_G^j}{j!}$$

since these are Poisson distributions with parameter λ_B or λ_G chosen with probability p or $1-p$ respectively. Finally,

$$p_{XY}(0,0)=pe^{-\lambda_B}+(1-p)e^{-\lambda_G}$$

Part b The probability mass function $p_X(i)$ is defined by $p_X(i)=\sum_j p_{XY}(i,j)$ For i nonzero,

$$p_X(i)=p_{XY}(i,0)=pe^{-\lambda_B}\frac{\lambda_B^i}{i!}$$

and

$$p_X(0)=p_{XY}(0,0)+\sum_{j=1}^{\infty}p_{XY}(0,j)=pe^{-\lambda_B}+(1-p)$$

since the total probability that the girls are chosen is $1-p$.

Similarly,

$$p_Y(j)=\begin{cases}(1-p)e^{-\lambda_G}\frac{\lambda_G^j}{j!} & j>0\\ (1-p)e^{-\lambda_G}+p & j=0\end{cases}$$

Part c Let $A=\{1\}$ and $B=\{1\}$. Then $\Pr[X\in A \,\text{and}\, Y\in B]=0$ since $p_{XY}(1,1)=0$. Also, $\Pr[X\in A]=p_X(1)=p\lambda_B e^{-\lambda_B}$ and $\Pr[Y\in B]=p_Y(1)=(1-p)\lambda_G e^{-\lambda_G}$. The product of these is clearly nonzero. So $\Pr[X\in A \,\text{and}\, Y\in B]=0\neq\Pr[X\in A]\Pr[Y\in B]$. Thus X and Y are not independent.

Part d The values of X are given by its probability mass function p_X. Since for all non-zero values of i, $p_X(i)$ is identical to the probability mass function of a Poisson random variable with parameter λ_B scaled by p, the expected value $\mathrm{E}[X]$ is simply obtained by scaling the expected value of the Poisson random variable accordingly:

$$\mathrm{E}[X]=p\lambda_B$$

and similarly

$$\mathrm{E}[Y]=(1-p)\lambda_G$$

By linearity of expectation

$$\mathrm{E}[X+Y]=\mathrm{E}[X]+\mathrm{E}[Y]=p\lambda_B+(1-p)\lambda_G$$

Part e For $m>0$, $X+Y$ can only equal m if $X=m$ or $Y=m$; no other cases are possible. So

$$p_{X+Y}(m)=p_X(m)+p_Y(m)=pe^{-\lambda_B}\frac{\lambda_B^m}{m!}+(1-p)e^{-\lambda_G}\frac{\lambda_G^m}{m!}$$

and if $m=0$

$$p_{X+Y}(0)=p_{XY}(0,0)=pe^{-\lambda_B}+(1-p)e^{-\lambda_G}$$

If $\lambda_B = \lambda_G = \lambda$ then the formula reduces to

$$p_{X+Y}(m) = [p + (1-p)]\left(e^{-\lambda}\frac{\lambda^m}{m!}\right) = e^{-\lambda}\frac{\lambda^m}{m!}$$

for $m \geq 0$ (the $m = 0$ case is also satisfied by this equation). This is the probability mass function of a Poisson distribution with parameter λ.

Part f The example differs from this problem in that, in the example, each of the day's customers can be a boy or girl, independent of the other. In this problem, all the customers on one day have the same gender. Since the total number of customers in each case is a Poisson random variable, and the proportions of each gender are the same, the expectations of X, Y, and $X + Y$ and the distributions of $X + Y$ are identical (assuming $\lambda_B = \lambda_G = \lambda$. But the distributions of X and Y are not independent in this problem. 40

Problem 3:

Part a The probability of success of each trial in this experiment is $\frac{4}{3\pi}$. So if the trial is performed 1000 times, the expected number of successes is $\frac{4000}{3\pi}$. If N successes are actually obtained, the value of π is approximately $\frac{4000}{3N}$. ✓

Part b The procedure performs 1000 trials with probability of success $p = \frac{4}{3\pi}$. We expect the number of successes to be given by a binomial distribution, so the variance is $1000p(1-p) = 1000\frac{4}{3\pi}(1 - \frac{4}{3\pi}) = \frac{4000(3\pi-4)}{9\pi^2} \approx 244.3$. So the standard deviation of N is $\sigma \approx \sqrt{244.3} \approx 15.63$. The binomial distribution approximates the normal distribution for large n, so we can say with over 95% confidence that N will be within two standard deviations, or 31.26 of the expected value $\frac{4000}{3\pi}$ (using the area Φ under the standard normal distribution). So N will probably be between $\frac{4000}{3\pi} - 31.26 \approx 393.154$ and $\frac{4000}{3\pi} + 31.26 \approx 455.673$. Thus the approximation for $\pi = \frac{4000}{3N}$ will be between $\frac{4000}{3\ 393.154} \approx 2.926$ and $\frac{4000}{3\ 455.673} \approx 3.391$, with just over 95% confidence. ✓

Part c Suppose we wish the error in the approximation to be not more than 0.0005, again with approximately 95% (2σ) confidence. Let X_i be Bernoulli random variables representing each trial, and $Y = \frac{\sum_{i=1}^{n} X_i}{n}$. Then $\mathrm{E}[Y] = \frac{4}{3\pi}$, and $\mathrm{Var}\, Y = \frac{n}{n^2} = \frac{\frac{4}{3\pi}(1-\frac{4}{3\pi})}{n} \approx \frac{.2443}{n}$. So $\sigma_Y = \sqrt{\mathrm{Var}\, Y} \approx \sqrt{\frac{.2443}{n}} \approx \frac{.4943}{\sqrt{n}}$. Again, with over 95% probability, the result will differ from the expected value by less than $2\sigma = \frac{.9886}{\sqrt{n}}$.

We want $3.1410 < \frac{4}{3Y} < 3.1420$, or equivalently $.424358 < Y < .42493$. Subtracting the mean $\frac{4}{3\pi}$, $-.000055 < Y - \mu_Y < .00008$. So we need the difference from the expected value, $2\sigma = \frac{.9886}{\sqrt{n}}$ to be less than .000055. n must be larger than 5.518×10^8. ✓

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