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18.440 **Dan Ports**

Problem Set 9

2003/11/19 drkp@mit.edu

Problem 1: Since X is distributed uniformly over $[-1,1]$, $f_X(x) = \frac{1}{2}$ for all x in this range and 0 otherwise. Therefore, the kth moment of X is

$$\begin{aligned} m_k(X) &= \mathrm{E}\left[X^k\right] = \int_{-1}^{1} \frac{1}{2} x^k \, dx \\ &= \left[\frac{1}{2}\frac{x^{k+1}}{k+1}\right]_{-1}^{1} \\ &= \begin{cases} 0 & k \text{ odd} \\ \frac{1}{k+1} & k \text{ even} \end{cases} \end{aligned}$$

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Problem 2: Let X and Y be random variables with moments $m_p(X)$ and $m_q(Y)$ defined. Then

$$\begin{aligned} m_k(X+Y) &= \mathrm{E}\left[(X+Y)^k\right] \\ &= \mathrm{E}\left[\sum_{i=0}^{k} \binom{k}{i} X^i Y^{k-i}\right] \quad \text{by the binomial theorem} \\ &= \sum_{i=0}^{k} \binom{k}{i} \mathrm{E}\left[X^i Y^{k-i}\right] \quad \text{by linearity of expectation} \\ &= \sum_{i=0}^{k} \binom{k}{i} \mathrm{E}\left[X^i\right] \mathrm{E}\left[Y^{k-i}\right] \quad \text{by independence of } X \text{ and } Y \\ &= \sum_{i=0}^{k} \binom{k}{i} m_i(X) m_{k-i}(Y) \quad \text{by definition} \end{aligned}$$

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Problem 3:

Part a

$$\mathrm{E}[S_n] = \mathrm{E}[X_1 + X_2 + \cdots + X_n] = \mathrm{E}[X_1] + \mathrm{E}[X_2] + \cdots + \mathrm{E}[X_n] = 0$$

√

Part b First, note that since $\mathrm{E}[X_i] = 0$, $\mathrm{Var}\, X_i = \mathrm{E}\left[X_i^2\right] = m_2(X_i) = \frac{1}{3}$. Since the variables are independent,

$$\mathrm{Var}\, S_n = \mathrm{Var}\, X_1 + X_2 + \cdots + X_n = \mathrm{Var}\, X_1 + \mathrm{Var}\, X_2 + \cdots + \mathrm{Var}\, X_n = \frac{n}{3}$$

√

Part c Consider first the case of even k. Then for odd i, $m_i(X) = m_{k-i} = 0$. Hence,

$$
\begin{aligned}
m_k(S_2) &= \sum_{i=0}^{k} \binom{k}{i} m_i(X_1) m_{k-i}(X_2) \\
&= \sum_{i=0 \text{ even}}^{k} \binom{k}{i} m_i(X_1) m_{k-i}(X_2) \\
&= \sum_{i=0}^{\frac{k}{2}} \binom{k}{i} \frac{1}{i+1} \frac{1}{k-i+1}
\end{aligned}
$$

$= \sum_{p=0}^{k/2} \binom{k}{2p} \frac{1}{2p+1} \frac{1}{2q-2p+1}$ o.k.

$i = 2p$

Performing a change of variables, with $p = i$ and $q = \frac{k}{2}$,

$$
\begin{aligned}
&= \sum_{p=0}^{q} \binom{2q}{2p} \frac{1}{2p+1} \frac{1}{2q-2p+1} \quad \checkmark \\
&= \sum_{p=0}^{q} \frac{(2q)!}{(2p)!(2q-2p)!} \frac{1}{2p+1} \frac{1}{2q-2p+1} \quad \checkmark \\
&= \sum_{p=0}^{q} \frac{(2q)!}{(2p+1)!(2q-2p+1)!} \quad \checkmark \\
&= \sum_{p=0}^{q} \binom{2q}{2p+1} = 2^{2q-1} = 2^{k-1}
\end{aligned}
$$

$= \frac{2^{k+1}}{(k+1)(k+2)}$

no, because $(2p+1) + 2q - 2p + 1 = 2q + 2 \neq 2q!$

Then, consider the case of k odd. Then if i is odd, $m_i(X) = 0$ and if i is even, $k - i$ is odd and $m_{k-i}(X) = 0$. Hence, each term in the sum is zero

$$
\begin{aligned}
m_k(S_2) &= \sum_{i=0}^{k} \binom{k}{i} m_i(X_1) m_{k-i}(X_2) \\
&= \sum_{i=0}^{k} 0 = 0
\end{aligned}
$$

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So

$$
m_k(X) = \begin{cases} 2^{k-1} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}
$$

Problem 4: From above,

$$
f_{S_1} = f_X = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}
$$

Performing the convolutions graphically (see the attached figures),

$$
\begin{aligned}
f_{S_2}(x) &= f_{S_1}(x) * f_{S_1}(x) \\
&= \int_{-\infty}^{\infty} f_{S_1}(t) f_{S_1}(x-t)\, dt \\
&= \begin{cases} \frac{x}{8} + \frac{1}{4} & -2 \le x \le 0 \\ \frac{1}{4} - \frac{x}{8} & 0 < x \le 2 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

$$
\begin{aligned}
f_{S_3}(x) &= f_{S_1}(x) * f_{S_1}(x) * f_{S_1}(x) \\
&= f_{S_2}(x) * f_{S_1}(x) \\
&= \int_{-\infty}^{\infty} f_{S_2}(t) f_{S_1}(x-t)\, dt \\
&= \begin{cases} \frac{1}{36}(x-3)(x+3) & -3 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

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Problem 5:

Part a Since $f(x) = \frac{1}{2}$ for x between -1 and 1 and 0 otherwise, $f(x-t) = \frac{1}{2}$ for t between $x-1$ and $x+1$ and 0 otherwise. Hence,

$$
\begin{aligned}
(f * f_X)(x) &= \int_{-\infty}^{\infty} f(t) f_X(x-t)\, dt \\
&= \int_{x-1}^{x+1} f(t) \frac{1}{2}\, dt \\
&= \frac{1}{2} \int_{x-1}^{x+1} f(t)\, dt
\end{aligned}
$$

Part b

$$
\begin{aligned}
(f * f_X)(x+\Delta x) - (f * f_X)(x) &= \frac{1}{2}\int_{x+\Delta x-1}^{x+\Delta x+1} f(t)\, dt - \frac{1}{2}\int_{x-1}^{x+1} f(t)\, dt \\
&= \frac{1}{2}\int_{x+\Delta x-1}^{x+1} f(t)\, dt + \frac{1}{2}\int_{x+1}^{x+\Delta x+1} f(t)\, dt \\
&\quad - \frac{1}{2}\int_{x-1}^{x+\Delta x-1} f(t)\, dt - \frac{1}{2}\int_{x+\Delta x-1}^{x+1} f(t)\, dt \\
&= \frac{1}{2}\int_{x+1}^{x+\Delta x+1} f(t)\, dt - \frac{1}{2}\int_{x-1}^{x+\Delta x-1} f(t)\, dt
\end{aligned}
$$

Part c

$$
\begin{aligned}
\frac{d}{dx}\left(f * f_X\right)(x) &= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\int_{x+\Delta x-1}^{x+\Delta x+1} f(t)\,dt - \frac{1}{2}\int_{x-1}^{x+1} f(t)\,dt}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\frac{1}{2}\int_{x+1}^{x+\Delta x+1} f(t)\,dt - \frac{1}{2}\int_{x-1}^{x+\Delta x-1} f(t)\,dt}{\Delta x}
\end{aligned}
$$

Noting that as Δx becomes small, the integrals can be approximated by their width multiplied by the value at one of the endpoints,

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{\frac{1}{2}f(x+1)\Delta x - \frac{1}{2}f(x-1)\Delta x}{\Delta x} \\
&= \frac{1}{2}f(x+1) - \frac{1}{2}f(x-1)
\end{aligned}
$$

Part d Assuming g is a function on $\mathbb{R}$ having a continuous derivative that vanishes outside some finite interval,

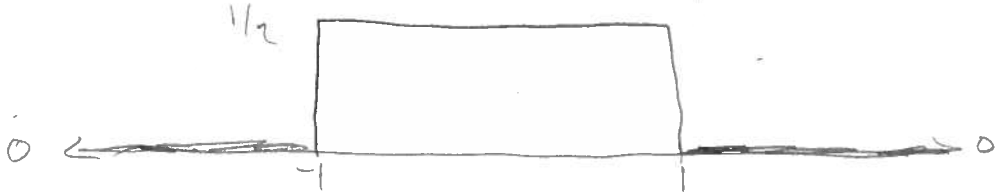
$$
\begin{aligned}
\frac{d}{dx}[f * g] &= \frac{d}{dx}\int_{-\infty}^{\infty} f(t)g(x-t)\,dt \\
&= \int_{-\infty}^{\infty} \frac{\partial}{\partial x} f(t)g(x-t)\,dt \\
&= \int_{-\infty}^{\infty} f(t)\frac{\partial}{\partial x} g(x-t)\,dt \\
&= \int_{-\infty}^{\infty} f(t)g'(x-t)\,dt \\
&= (f * g')(x)
\end{aligned}
$$

Noting that the function f_X is a square pulse, its derivative f'_X is a pair of impulses: a positive impulse of magnitude $\frac{1}{2}$ at $x = -1$ and a negative impulse of the same magnitude at $x = 1$. Since the convolution of an impulse with a function is simply the function shifted and scaled, the convolution with the derivative of f_X is the sum of two copies of the function scaled by $\frac{1}{2}$, one positive and shifted a unit to the left, and another negative and shifted to the right. The formula from part c follows:

$$
\frac{d}{dx}\left(f * f_X\right)(x) = \frac{1}{2}f(x+1) - \frac{1}{2}f(x-1)
$$

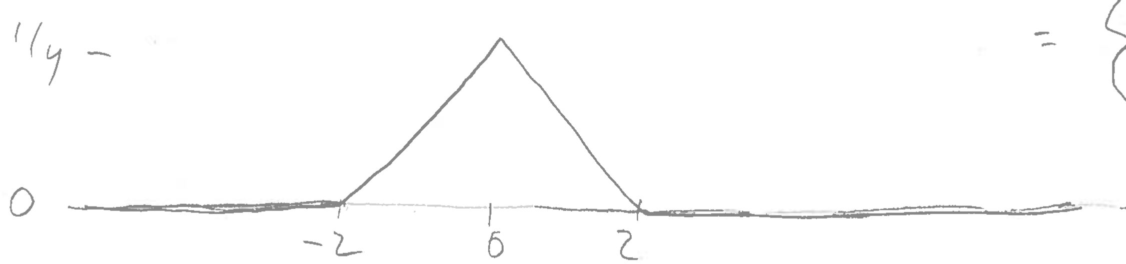
Figures for problem 4

a $\quad f_X = \begin{cases} 1/2 & x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$ $\quad$ o.w

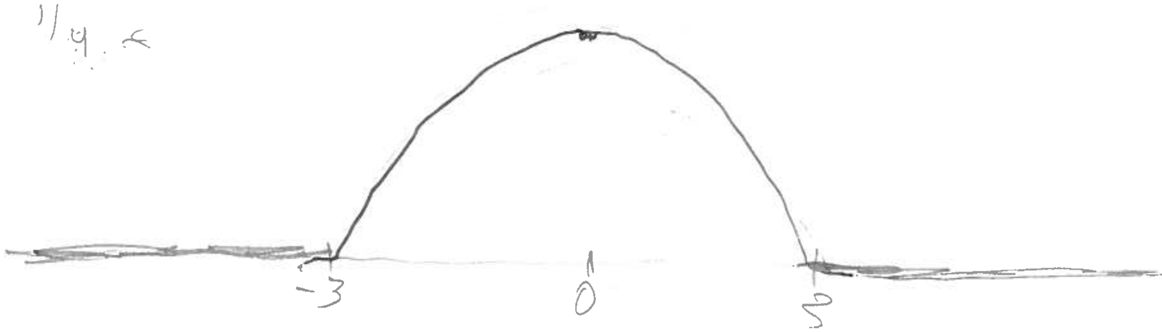


b. The convolution of two square pulses f_X is a triangle with vertices $(-2,0)$, $(2,0)$, and $(0, 1/4)$

$$f_X * f_X = \begin{cases} X/8 + 1/4 & x \in [-2,0] \\ 1/4 - X/8 & x \in [0,2] \\ 0 & \text{otherwise} \end{cases}$$



c. The convolution of the triangle with a square pulse is a parabola. The parabola has x-intercept at ± 3 and area 1:



$$f_X * f_X * f_X = \begin{cases} -\frac{1}{36}(x-3)(x+3) & x \in [-3,3] \\ 0 & \text{otherwise} \end{cases}$$