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18.440 Exam 1
Friday, October 10, 2003

Show all your work on these sheets; you can write on the backs of sheets if you need more space.

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1. **(20 points)** You have three fair six-sided dice (red, green, and blue). You perform the experiment of rolling the dice.

a) Describe a reasonable sample space S for this experiment, and describe the probability function P on events in S.

Let S be the set of ordered 3-tuples $\langle i, j, k \rangle$, w/ i, j, k each between 1 and 6. i is the value of the red die, j green, k blue. For an event E, define

$P(E) = \frac{|E|}{|S|} = \frac{|E|}{6^3}$ since each outcome is equally likely.

b) Let E be the event that the red and green dice add up to 7, and let F be the event that the green and blue dice add up to 5. Show that E and F are independent.

By enumeration. E: $\langle 1,6,n \rangle$, $\langle 2,5,n \rangle$, $\langle 3,4,n \rangle$, $\langle 4,3,n \rangle$, $\langle 5,2,n \rangle$, $\langle 6,1,n \rangle$

F: $\langle n,1,4 \rangle$, $\langle n,2,3 \rangle$, $\langle n,3,2 \rangle$, $\langle n,4,1 \rangle$

EF: $\langle 3,4,1 \rangle$, $\langle 4,3,2 \rangle$, $\langle 5,2,3 \rangle$, $\langle 6,1,4 \rangle$

where n is any integer from 0 to 6 (over)

c) Let X be the random variable giving the sum of the red and green dice, and Y the random variable giving the sum of the green and blue dice. Are X and Y independent? (To say that discrete random variables are independent means that for every value a of X and b of Y, the events $X = a$ and $Y = b$ are independent.)

No. If no information is available, the probability that $X = 12$ is $\frac{1}{6} \cdot \frac{1}{6}$. But if $Y = 2$, then the green die must be a 1 and so the sum of the red and green dice cannot be 12.

$P(X=12 \mid Y=2) = 0 \neq \frac{1}{36} = P(X=12)$. Not independent.

d) Calculate the expected value of X.

Let A and B be the value of the red and green dice respectively. $E(A) = E(B) = \sum_{i=1}^{6} i/6 = 3.5$.

$X = A + B$ so by linearity of expectation

$E[X] = E[A] + E[B] = 3.5 + 3.5 = 7$

16. $|E| = 6 \cdot 6 = 36 \Rightarrow P(E) = \frac{36}{6^3}$

$|F| = 4 \cdot 6 = 24 \Rightarrow P(F) = \frac{24}{6^3}$

$|EF| = 4 \Rightarrow P(EF) = \frac{4}{6^3} = \frac{36 \cdot 24}{6^6} = P(E)P(F)$

So E, F are independent.

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2. (30 points) An urn contains n balls, numbered 1 through n. A ball is removed at random, its number noted, and the ball replaced. Repeating m times, provides m random integers $(x_1, \ldots, x_m)$, each between 1 and n. Let X be the largest of the x_i, a random variable taking values between 1 and n.

a) Describe a reasonable sample space S for this experiment, and describe the probability function P on events in S.

Let S be the set of ordered m-tuples with values 1 to n, representing the number of each ball drawn in order. Define $P(E) = \frac{|E|}{|S|} = \frac{|E|}{n^m}$

b) Compute the probability of the event $X \leq k$ (for k between 1 and n).

This requires that every ball drawn be one of the k ones with value $\leq k$. So there are k^m possible outcomes, and $P(X \leq k) = k^m / n^m$

c) Compute $P(X = k)$.

$P(X = k) = P(X \leq k) - P(X \leq k-1) = \frac{k^m - (k-1)^m}{n^m}$

d) Compute exactly the expected value of X in the case $m = 1$.

$E[X] = \sum_{i=1}^{n} i \frac{i-(i-1)}{n} = \sum_{i=1}^{n} \frac{i}{n} = \frac{1}{n}\sum_{i=1}^{n} i = \frac{1}{n}\left(\frac{n(n+1)}{2}\right) = \frac{(n+1)}{2}$

e) Compute an approximation to the expected value of X for general m, using the formula

$$\sum_{j=0}^{n-1} j^m \approx n^{m+1}/(m+1).$$

(Hint: it's helpful to write $k(k-1)^m = (k-1)^{m+1} + (k-1)^m$.)

$\cancel{E[X] = \sum_{i=1}^{n} i \frac{i^m - (i-1)^m}{n^m} = \frac{1}{n^m} \sum}$

on back

f) Explain what ought to happen to $E(X)$ as m tends to infinity. Does your formula in (e) conform to this?

As $m \to \infty$, it is increasingly likely that we draw the nth ball, so we expect $E(X) \to n$. In the formula from part e,

$\lim_{m\to\infty} \frac{1}{n^m}\left[n^{m+1} - \frac{n^{m+1}}{m+1}\right] = \lim_{m\to\infty} \frac{n^{m+1}}{n^m}\left[1 - \frac{1}{m+1}\right]$

$= \lim_{n\to\infty} \frac{n^{m+1}}{n^m} = n$, as expected.

2e.

$$E[X] = \sum_{i=1}^{n} i \cdot \frac{i^m - (i-1)^m}{n^m} = \frac{1}{n^m} \sum_{i=1}^{n} i^{m+1} - i(i-1)^m$$

$$= \frac{1}{n^m} \sum_{i=1}^{n} i^{m+1} - (i-1)^{m+1} - (i-1)^m$$

$$= \frac{1}{n^m} \Bigg[\underbrace{\sum_{i=1}^{n} i^{m+1} - (i-1)^{m+1}}_{\text{telescoping sum}} - \sum_{i=1}^{n} (i-1)^m \Bigg]$$

$$= \frac{1}{n^m} \Big[(n^{m+1} - 0^{m+1}) - \sum_{j=0}^{n-1} j^m$$

$$\boxed{\approx \frac{1}{n^m} \left[n^{m+1} - \frac{n^{m+1}}{m+1} \right]}$$

$$= n \cdot \frac{m}{m+1}$$

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3. (20 points) By a "five-digit number" I mean a string of five digits, each of which is allowed to be $0, 1, \ldots, 9$; an example of a five-digit number is 00797. Suppose a five-digit number is chosen at random, with all outcomes equally likely.

a) What is the probability that no pair of consecutive digits is the same? (Computational assistance: $81^2 = 6561$.)

There are 10^5 possible 5-digit numbers. To find a 5-digit number with no two consecutive digits alike, there are 10 choices for the first number, and 9 each for the others. So $P(x) = \frac{10 \cdot 9^4}{10^5} = .6561$

b) What is the probability that the number is a palindrome? (A palindrome reads the same forwards as backwards, like 17871.)

In a palindrome, the last two digits are determined by the first two. So we can only choose the first 3 numbers to form a palindrome.
$P(y) = 10^3/10^5 = 1/100$

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4. (20 points) Suppose that X is a binomial random variable with parameters n and p, so that

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

Calculate the expected value of X. (You may know the answer. Just writing it down is not enough; you need to show how to arrive at it from the definition, explaining what you're doing at each step.)

On back.

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5. (10 points) The number of exams in an MIT course is a Poisson random variable averaging 3. (Helpful hints: $e^{-3} = .0498$, and $(1 - e^{-3})^{30} = .216$.)

Call the variable X.

a) What's the probability that a particular course has at least one exam?

$P(X \geq 1) = 1 - P(X=0) = 1 - \frac{e^{-3} \cdot 3^0}{0!} = 1 - e^{-3} \approx .950$

b) If you take thirty courses at MIT, what's the probability that you'll have at least one course with no exams?

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This is the probability of having ~~no~~ at least one courses with no exams, which is 1 minus the prob. of having 30 classes w/ at least 1 exam. From above, the probability of having at least 1 exam in a class is $1 - e^{-3}$. So the probability we are interested in is $1 - (1 - e^{-3})^{30} = 1 - .216 = .784$

5. $P(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$ is given.

$E[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k}$ by definition

Note that

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$$

by the binomial theorem

$$E[X] = \sum_{k=0}^{n} k \frac{n!}{(n-k)!\,k!} p^k (1-p)^{n-k} = \sum_{k=0}^{n} \frac{n!}{(n-k)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \frac{n!\,(n-k+1)}{(n-k+1)!\,(k-1)!} p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k-1} (n-k+1)\, p^k (1-p)^{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k-1} (n-k+1) \frac{p}{1-p} p^{k-1} (1-p)^{n-k+1}$$

$$= \frac{p}{1-p} \sum_{k=0}^{n}$$

⋮

$$E[X] = np$$