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18.440 Exam 2
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Show all your work on these sheets; you can write on the backs of sheets if you need more space. Calculators are permitted but shouldn't be needed.

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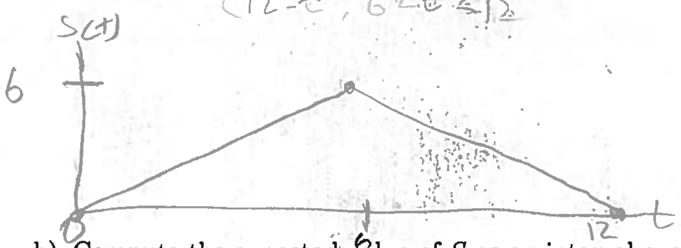
1. (30 points) A point t is selected at random from the interval $[0, 12]$ (with a uniform distribution, meaning that the probability of a subinterval is equal to its length divided by 12). You can think of t as a point on a twelve-inch ruler. The interval is divided into two segments by this point. Define $S(t)$ to be the length of the shorter segment.

a) Sketch a graph of $S(t)$ on the interval $[0, 12]$.

$$S(t) = \begin{cases} t, & 0 \le t \le 6 \\ 12 - t, & 6 < t \le 12 \end{cases}$$



✓

b) Compute the expected value of S as an integral over $[0, 12]$.

$$E(S) = \int_0^{12} \frac{1}{12} S(t) = \frac{1}{12}\int_0^6 t\,dt + \frac{1}{12}\int_6^{12} 12 - t\,dt$$

$$= \frac{1}{12}\left[\frac{t^2}{2}\Big|_0^6 + 12t - \frac{t^2}{2}\Big|_6^{12}\right]$$

$$\frac{1}{12}\left[18 + 18\right] = \frac{36}{12} = \boxed{3}$$ ✓

c) Compute the cumulative distribution function $F_S(x) = P(S \le x)$, and the probability density $f_S(x)$.

$$F_S(x) = P(S \le x) = P(t \le x \cup 12 - t \le x) = P(t \le x \cup t \ge 12 - x)$$

$$= \frac{1}{12}\left[(x - 0) + (12 - (12 - x))\right] = \frac{2x}{12} = \frac{x}{6} \checkmark \text{ on } 0 \le x \le 6$$

(over)

d) Use the probability density function to compute $E(S^2)$.

✓ $$E(S^2) = \int s^2 f_S(s)\,ds = \int_0^6 s^2 \frac{1}{6}\,ds = \frac{x^3}{18}\Big|_0^6$$

$$= \boxed{12}$$

1c

$$F_S(x) = \begin{cases} 1 & x \geq 6 \\ \frac{x}{6} & 0 \leq x < 6 \\ 0 & x < 0 \end{cases}$$

$f_S(x) = \frac{d}{dx} F_S(x) = \frac{1}{6}$ ✓ on $0 \leq x < 6$

and 0 otherwise

34 **2. (35 points)** The number of snowstorms occuring during the first t days of winter is a Poisson random variable N_t with parameter $t\lambda$. (In Boston, $\lambda = 1/10$ might be a reasonable model.) Suppose that the first storm appears after S_1 days, the second storm after S_2 days, and so on. Each S_m is a random variable taking values in $[0, \infty)$.

a) Write a formula for $P(N_t = k)$, the the probability of having exactly k storms in the first t days of winter. (The formula should be a function of k, t, and λ.)

~~N_t is a Poisson RV,~~ so

$$\boxed{P(N_t=k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}}$$ ✓

b) Write a formula for $P(S_2 \le t)$, the probability that at least two storms occur in the first t days of winter. (There is a correct formula involving an infinite sum, but it's better to find a formula that's a finite sum.)

This is the probability $P(N_t \neq 0 \text{ or } 1)$

$= 1 - P(N_t=0) - P(N_t=1)$

$$\boxed{= 1 - e^{-\lambda t} - \lambda t e^{-\lambda t}}$$ ✓

\ c) Compute the probability density function $f_{S_2}(t)$ for the random variable S_2.

S_2 is the expected arrival time of the second storm.

This is a gamma RV w/ parameters λ and $\alpha = 2$

$$f_{S_2}(t) = \frac{\lambda e^{-\lambda t}(\lambda t)^{\alpha-1}}{\Gamma(\alpha)} = \boxed{\lambda e^{-\lambda t}\cdot\lambda t \quad \begin{array}{ll} & \text{for } t>0 \\ 0 & t<0 \end{array}}$$

d) Write an integral formula for the expected value of S_2, the expected arrival time for the second storm.

$$E(S_2) = \int_0^\infty t f_{S_2}(t)\,dt = \int_0^\infty t\lambda e^{-\lambda t}\lambda t\,dt = \boxed{\int_0^\infty (\lambda t)^2 e^{-\lambda t}}$$ ✓

$$= \frac{\alpha}{\lambda} = \boxed{2/\lambda}$$ (by expectation of a gamma RV)

e) Using $\lambda = 1/10$, compute $E(S_2)$ as a number.

✓ $$E(S_2) = \int_0^\infty \left(\frac{t}{10}\right)^2 e^{-\frac{1}{10}t} = \left[\frac{-(t^2+20t+200)}{10}e^{-t/10}\right]_0^\infty$$

$$= 0 - -\left(\frac{0^2+20\cdot 0+200}{10}\right)e^{-0} = \frac{200}{10} = 20$$

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3. (35 points) A table of values of

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\,dt$$

appears below! ~~on a separate page~~.

Suppose X is a binomial random variable with $n = 100$ and $p = 1/10$.

a) What is the expected value of X? (It's enough to quote the general formula and compute the number.)

$E(X) = np = \frac{100}{10} = \boxed{10}$

b) What is the standard deviation of X? (Again, quote a formula and compute a number.)

$\text{Var } X = n(p)(1-p) = 100 \cdot 1/10 \cdot 9/10 = 9$

$\sigma^2 = \text{Var } X.$ Std dev $X = \sigma = \sqrt{\text{Var } X} = \sqrt{9} = \boxed{3}$

c) Use the normal approximation and the accompanying table to estimate the probability that $9 \leq X \leq 14$. Don't forget to include the "continuity correction."

-3

$P(9 \leq X \leq 14) = P\left(\frac{9-\mu}{\sigma} \leq \frac{X-\mu}{\sigma} \leq \frac{14-\mu}{\sigma}\right)$

$= P\left(\frac{9-10}{3} \leq \frac{X-\mu}{\sigma} \leq \frac{14-10}{3}\right) = P\left(-1/3 \leq \frac{X-\mu}{\sigma} \leq 4/3\right)$

$\frac{X-\mu}{\sigma}$ is a standard normal random var. approximately

So

$P(9 \leq X \leq 14)$

$\approx \Phi(4/3) - \Phi(-1/3)$

$= \Phi(4/3) - (1 - \Phi(1/3))$

$\approx \Phi(1.33) + \Phi(0.33) - 1$

$\approx .9082 + .6293 - 1$

$= 0.5375$

(over)

TABLE 5.1 AREA $\Phi(x)$ UNDER THE STANDARD NORMAL CURVE TO THE LEFT OF x

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

3c continued.

A better approximation using the continuity of $\Phi(x)$ is

$$P(9 \le X \le 14) \approx \Phi(4/3) + \Phi(1/3) - 1$$

$$\approx \Phi(1.33) + \frac{1}{3}\left[\Phi(1.34) - \Phi(1.33)\right]$$
$$+ \Phi(.33) + \frac{1}{3}\left[\Phi(0.34) - \Phi(.33)\right] - 1$$

$$= .6293 + 1/3(.6331 - .6293)$$
$$+ .9082 + 1/3(.9099 - .9082) - 1$$

$$\boxed{= .53933}$$

This is using the differentiability of Φ. It's correct, but it isn't the continuity correction.