8/10 ~~28~~

Dan Ports
6.002 PS 10

1. A. $i_s = i_C + i_L = C\frac{dv_C}{dt} + i_L = LC\frac{d^2 i_L}{dt^2} + i_L$

$\frac{d^2 i_L}{dt^2} + \frac{i_L}{LC} = \frac{I_s}{LC} \Rightarrow s^2 + \frac{1}{LC} = 0 \Rightarrow s = \pm\frac{1}{\sqrt{LC}}\, i$

$i_{Lp} = I_s$

$i_{LH} = A\cos(\frac{1}{\sqrt{LC}}t) + B\sin(\frac{1}{\sqrt{LC}}t)$

$i_L(0) = I_s + A = 0 \Rightarrow A = -I_s$

$v_C(0^-) = L\frac{di_L}{dt}\Big|_0 = B = 0 \Rightarrow B = 0$

$\boxed{i_L(t) = I_s - I_s\cos(\frac{1}{\sqrt{LC}}t) \quad \text{for } t>0}$

b. $i_{Lp} = 0$

$i_{LH} = A\cos(\frac{1}{\sqrt{LC}}t) + B\sin(\frac{1}{\sqrt{LC}}t)$

$i_L(0) = A \Rightarrow A = I_L$

$v_C(0) = L\frac{di_L}{dt}\Big|_0 = B = 0 \Rightarrow B = 0$

$\boxed{i_L(t) = I_L\cos(1/\sqrt{LC}\, t)}$

c. $i_{Lp} = I_s$

$i_{LH} = A\cos(1/\sqrt{LC}\, t) + B\sin(1/\sqrt{LC}\, t)$

$i_L(0) = I_s + A = I_L \Rightarrow A = I_L - I_s$

$v_C(0) = L\frac{di_L}{dt}\Big|_0 = B = 0 \Rightarrow B = 0$

$\boxed{i_L(t) = I_s + (I_L - I_s)\cos(\frac{1}{\sqrt{LC}}t)}$

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2. $v_0(t) = v_s(t) - v_c(t) \Rightarrow v_c(t) = v_s(t) - v_0(t)$

$i_{R_1}(t) + i_C(t) = i_{R_2}$

$\Rightarrow v_c(t)/R_1 + C\frac{dv_c}{dt} = \frac{v_0(t)}{R_2}$

$\Rightarrow \frac{v_s(t)-v_0(t)}{R_1} + C\frac{d}{dt}[v_s(t)-v_0(t)] = \frac{v_0(t)}{R_2}$

$$\Rightarrow \boxed{\frac{v_s(t)}{R_1} + C\frac{dv_s(t)}{dt} = v_0(t)\left[\frac{1}{R_1}+\frac{1}{R_2}\right] + C\frac{dv_0(t)}{dt}}$$

b. $\frac{V_s e^{j\omega t}}{R_1} + j\omega C V_s e^{j\omega t} = v_{0f}(t)\left[\frac{1}{R_1}+\frac{1}{R_2}\right] + C\frac{dv_{0f}(t)}{dt}$

$\frac{V_s}{R_1}e^{j\omega t} + j\omega C V_s e^{j\omega t} = \hat{V}_0 e^{j\omega t}\left[\frac{1}{R_1}+\frac{1}{R_2}\right] + j\omega C \hat{V}_0 e^{j\omega t}$

$V_s\left(\frac{1}{R_1} + j\omega C\right) = \hat{V}_0\left(\frac{1}{R_1}+\frac{1}{R_2}+j\omega C\right)$

$\hat{V}_0 = V_s\frac{1/R_1 + j\omega C}{1/R_1 + 1/R_2 + j\omega C} = V_s\frac{R_2 + R_1R_2 j\omega C}{R_2 + R_1 + R_1R_2 j\omega C}$

$$\boxed{v_{0f}(t) = \hat{V}_0 e^{j\omega t} = V_s\left[\frac{R_2 + R_1R_2 j\omega C}{R_2+R_1+R_1R_2 j\omega C}\right]e^{j\omega t}}$$

c. $V_s\cos(\omega t) = \mathrm{Re}(V_s e^{j\omega t})$,

so $v_0(t) = \mathrm{Re}(v_{0f}(t))$

$v_{0f}(t) = V_s\frac{R_2\sqrt{1+R_1^2\omega^2C^2}\; e^{j\tan^{-1}(\omega CR_1)}}{\sqrt{(R_2+R_1)^2+R_1^2R_2^2\omega^2C^2}\; e^{j\tan^{-1}\left(\frac{R_1R_2\omega C}{R_2+R_1}\right)}}e^{j\omega t}$

$= \frac{V_s\sqrt{1+R_1^2\omega^2C^2}}{\sqrt{(R_1+R_2)^2+R_1^2R_2^2\omega^2C^2}}e^{j\left(\omega t + \tan^{-1}(\omega CR_1) - \tan^{-1}\left(\frac{R_1R_2\omega C}{R_2+R_1}\right)\right)}$

$v_0(t) = \mathrm{Re}(v_{0f}(t))$

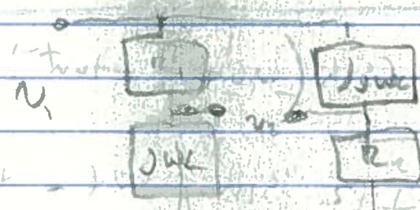
$$\boxed{v_0(t) = \frac{R_2V_s\sqrt{1+R_1^2\omega^2C^2}}{\sqrt{(R_1+R_2)^2+R_1^2R_2^2\omega^2C^2}}\cos(\omega t+\varphi) \quad \text{where } \varphi = \tan^{-1}(\omega CR_1) - \tan^{-1}\left(\frac{R_1R_2\omega C}{R_1+R_2}\right)}$$

3. a. i. $\dfrac{\frac{R}{j\omega C}}{R+\frac{1}{j\omega C}} = \dfrac{R}{j\omega C R+1}$ by parallel com

ii. $Y = \dfrac{\frac{1}{R_1} \cdot \frac{1}{j\omega L}}{1/R_1 + 1/j\omega L} + \dfrac{1}{R_2} + j\omega C$ by series/parallel combination

iii. $\dfrac{I_2}{I_1} = \dfrac{j\omega L + R}{j\omega L + R + 1/j\omega C}$ by current divider rel

iv. V_1



$V_2 = V_1 \dfrac{j\omega L}{R_1 + j\omega L} - V_1 \dfrac{R_2}{1/j\omega C + R_2}$

by voltage dividers

$$H = \frac{V_2}{V_1} = \frac{j\omega L}{R_1 + j\omega L} - \frac{R_2}{1/j\omega C + R_2}$$

B. $i_1(t) = I_0 \cos \omega t = Re(I e^{j\omega t})$

so $i_2(t) = Re(H(j\omega) I e^{j\omega t})$

$$= Re\left[\frac{j\omega L + R}{j\omega L + R + 1/j\omega C} I e^{j\omega t}\right]$$

$$= Re\left[\frac{\sqrt{R^2+(\omega L)^2}\, e^{j\tan^{-1}\left(\frac{\omega L}{R}\right)}}{\sqrt{R^2+(\omega L - 1/\omega C)^2}\, e^{j\tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)}} \cdot I e^{j\omega t}\right]$$

$$= I\sqrt{\frac{R^2+(\omega L)^2}{R^2+(\omega L - 1/\omega C)^2}} \cdot \cos(\omega t + \varphi)$$

$$\varphi = \tan^{-1}\left(\frac{\omega L}{R}\right) - \tan^{-1}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

c. $v_1(t) = V\sin(\omega t) = \mathrm{Im}(Ve^{j\omega t})$

So $v_2(t) = \mathrm{Im}(H(j\omega)\,Ve^{j\omega t})$

$$= \mathrm{Im}\left[\left(\frac{j\omega L}{R_1 + j\omega L} - \frac{R_2}{R_2 + 1/j\omega C}\right) Ve^{j\omega t}\right]$$

$$= \mathrm{Im}\left[\left(\sqrt{\frac{(\omega L)^2}{R_1^2 + (\omega L)^2}}\, e^{j\left(\frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R_1}\right)\right)} - \sqrt{\frac{R_2^2}{R_2^2 + \left(\frac{1}{\omega C}\right)^2}}\, e^{j\left(-\tan^{-1}\left(\frac{1/\omega C}{R_2}\right)\right)}\right) \cdot \left(Ve^{j\omega t}\right)\right]$$

1)

$$\boxed{v_2 = \left[V\sqrt{\frac{(\omega L)^2}{R_1^2 + (\omega L)^2}}\sin\left(\omega t + \frac{\pi}{2} - \tan^{-1}\left(\frac{\omega L}{R_1}\right)\right) - V\sqrt{\frac{R_2^2}{R_2^2 + \left(\frac{1}{\omega C}\right)^2}}\sin\left(\omega t - \tan^{-1}\left(\frac{1}{\omega C R_2}\right)\right)\right]}$$

$v_2(t) = 0$ since $L = R_1 R_2 C$

9a) $\boxed{v_t = v = 10\,V \sin(10^4 t)}$

$$Z_t = \frac{-10\,V}{5\,mA\; e^{-j\pi/6}} = -2\,k\Omega\; e^{j\pi/6}$$

$$\boxed{Z_t = -2\,k\Omega\left[\frac{\sqrt{3}}{2} + j\frac{1}{2}\right]}$$

9)

$$\hat{i} = \frac{-10\,V e^{j\omega t}}{Z_t + \frac{1}{j\omega C}} = \frac{-10\,V\, e^{j10^4 t}}{-\sqrt{3}\,k\Omega - j/2\,k\Omega + 2j\,k\Omega} = \frac{-10\,V}{-\sqrt{3} - \sqrt{3}/2\, j}\, e^{j10^4 t}$$

$$= \frac{10}{\sqrt{3}}\left(\frac{1}{1 + \sqrt{3}}\right) e^{j10^4 t} = \frac{10}{\sqrt{3}} \cdot \frac{1}{2}\, e^{-\arctan\sqrt{3}}\, e^{j10^4 t}$$

$$= \frac{5}{\sqrt{3}}\, e^{j10^4 t - \pi/3}$$

$$\boxed{i(t) = \mathrm{Im}(\hat{i}) = \frac{5}{\sqrt{3}}\sin(10^4 t - \pi/3)\ mA}$$

5.a. $P = i^2 R = \left(\frac{V_s}{R_s + R_L}\right)^2 \cdot R_L = \boxed{\frac{V_s^2 R_L}{(R_s + R_L)^2}}$

$\frac{dP}{dR_L} = \frac{-(R_L - R_s) V_s^2}{(R_s + R_L)^3}$. This equals 0 when $R_s = R_L$,

so P is maximized at $R_s = R_L$

At $R_s = R_L$, $P = \frac{V_s^2 R_L}{(R_L + R_L)^2} = \frac{V_s^2 R_L}{4 R_L^2} = \boxed{\frac{V_s^2}{4 R_L}}$

b. $\bar{P} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} R\left(\hat{I} \cdot e^{j\omega t}\right)^2 dt = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} |I|^2 \cos^2(\omega t)$

$= \frac{\omega}{2\pi} R |I|^2 \int_0^{2\pi/\omega} \cos^2(\omega t)\, dt = \frac{\omega R |I|^2}{2\pi} \int_0^{2\pi/\omega} \frac{1 + \cos(2\omega t)}{2}$

$= \frac{\omega R |I|^2}{2\pi} \cdot \frac{1}{2} \left[t - \frac{\sin(2\omega t)}{2\omega} \right]_{t=0}^{t = 2\pi/\omega}$

$= \frac{\omega R |I|^2}{2\pi} \cdot \frac{1}{2} \left[\frac{2\pi}{\omega} - (0 - 0) \right] = \boxed{\frac{|I|^2 R}{2}}$

c. $\hat{I} = \frac{V_s}{Z_L + R_s} = \frac{V_s}{R_s + R_L + X_L} \Rightarrow |I| = \frac{V_s}{\sqrt{(R_s + R_L)^2 + X_L^2}}$

$\bar{P} = |I|^2 R_L / 2 = \frac{V_s^2 R_L}{2(R_s + R_L)^2 + X_L^2}$

X_L must be zero to minimize the denominator and maximize the power. With $X_L = 0$,

$\bar{P} = \frac{V_s^2 R_L}{2(R_s + R_L)^2}$. From part a, we know this is maximized when $R_s = R_L$

d. i. $Z = \frac{R_L / j\omega C}{R_L + 1/j\omega C} = \boxed{\frac{R_L}{(R_L \omega C)^2 + 1} - \frac{R_L^2 \omega C}{(R_L \omega C)^2 + 1} j}$

ii. $\frac{R_L}{(R_L \omega C)^2 + 1} = R_s \Rightarrow R_L = R_s\left((R_L \omega C)^2 + 1\right)$

$\Rightarrow R_L - R_s = (R_L \omega C)^2 \Rightarrow \frac{R_L - R_s}{R_L^2 R_s} = (\omega C)^2$

$\Rightarrow \boxed{\omega C = \frac{1}{R_L} \cdot \sqrt{\frac{R_L}{R_s} - 1}}$

$$0 = j\omega L - j\frac{(R_L)^2 \omega C}{(R_L \omega C)^2 + 1}$$

$$\omega_L = \frac{R_L^2 \cdot \frac{1}{R_L}\sqrt{\frac{R_L}{R_S} - 1}}{R_L^2 \cdot \frac{1}{R_L^2}\left(\frac{R_L}{R_S} - 1\right) + 1}$$

$$\omega_L = \frac{R_L\sqrt{R_L/R_S - 1}}{R_L/R_S} = \boxed{R_S\sqrt{R_L/R_S - 1}}$$

For $R_L < R_S$, change the roles of the inductor and capacitor; let the inductor set the real part of the impedance to R_S and let the capacitor adjust the reactance to zero.

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