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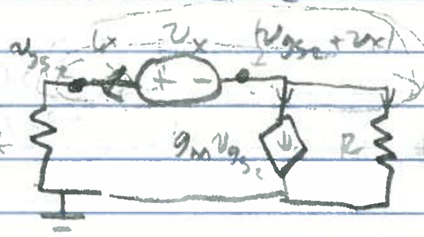
Dan Ports
6.002 PS7

1a. To find R_{Th}, disable all independent sources, apply a test voltage v_x and find the current through the terminals i_x



Applying nodal analysis on the supernode containing v_x:

$$\frac{v_{gs_2}}{R_t} + g_m v_{gs_2} + \frac{(v_{gs_2} - v_x)}{R} = 0$$

$$\Rightarrow \frac{v_{gs_2}}{R_t} + g_m v_{gs_2} + v_{gs_2}/R = v_x/R$$

$$\Rightarrow v_{gs_2}\left(1/R_t + g_m + 1/R\right) = \frac{v_x}{R}$$

$$v_{gs_2} = \frac{v_x}{R\left(1/R_t + g_m + 1/R\right)}$$

$$i_x = \frac{v_{gs_2}}{R_t} \Rightarrow \frac{v_x}{i_x} = \frac{v_x R_t}{v_{gs_2}}$$

$$\Rightarrow \frac{v_x}{i_x} = \frac{v_x R_t R\left(1/R_t + g_m + 1/R\right)}{v_x}$$

$$\frac{v_x}{i_x} = R_t R\left(1/R_t + g_m + 1/R\right)$$

$$\boxed{R_{Th} = \frac{v_x}{i_x} = (R + g_m R_t R + R_t)}$$

b. $\tau = \boxed{R_{Th} C_{GD}}$ w/ R_{Th} as above

$$\tau = (R + g_m R_t R + R_t)\, C_{GD}$$

c. $v_{out}(0) = v_{out}(0^-) = 0$ because v_t is 0 and there is no voltage or current flow in the

1. As $t \to \infty$, The capacitor is an open circuit. Then $V_{gs2}(\infty) = V_C$.
The current flowing through the resistor is provided by the current source: $i_R = -g_m v_{gs2} = -g_m v_t$
Thus $V_{out} = \cancel{i_R R}$ $-g_m v_{gs2} R$

The time constant is τ, so:
$$V_{out}(t) = -g_m v_C R \left(1 - e^{-t/\tau}\right)$$
w/ τ as above.

1.2 a. The capacitor is initially uncharged, $v(0) = 0$
Its maximum voltage at large t is V_0/R

Thus $v(t) = V_0 \left(1 - e^{-t/\tau}\right)$
Here, $\tau = RC$

b. The capacitor has a final voltage at large t of V_0. The energy in the capacitor at this time is $\frac{1}{2} C V^2 = \boxed{\frac{1}{2} C V_0^2}$

The voltage source outputs voltage V_0 and current $\frac{V_0}{R} e^{-t/RC}$ ~~Integrating~~, W all time, $V_0 \cdot V_0/R \cdot RC \cdot e^{-t/RC} \Big|_0^{\infty} = \boxed{C V_0^2}$

The difference in energy $\boxed{\frac{1}{2} C V_0^2}$ is dissipated in the resistor.

c. This system has initial current $i(0) = V_1/R$ because capacitor C_1 is a short. There is ~~no~~ current at $t = \infty$. The time constant of the system is $\tau = RC_{eq} = R \frac{C_1 C}{C_1 + C}$

Thus $i(t) = V_1/R \, e^{-t/\tau}$

$v(t) = 1/C_1 \int_0^t i(t) = V_1/RC_1 \int_0^t e^{-t/\tau}$

$$\boxed{v(t) = \frac{V_1}{RC_1} \tau \left(1 - e^{-t/\tau}\right) = \frac{V_1 C_{eq}}{C}\left(1 - e^{-t/\tau}\right)}$$

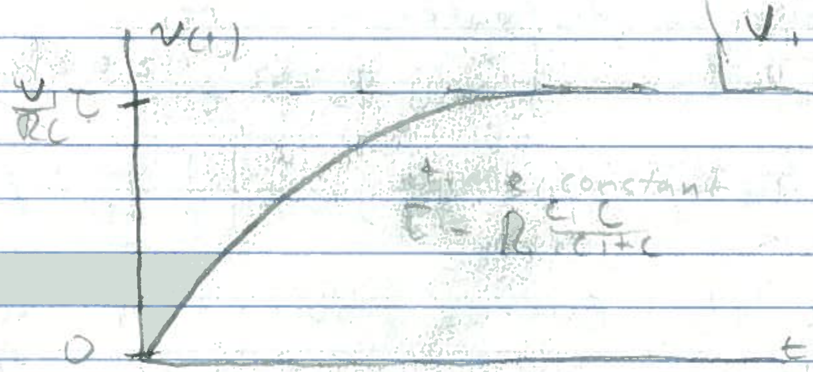
To have the same energy in the capacitor, as $t \to \infty$ $v(t)$ must equal V_0.

Thus $v(\infty) = V_0$

$$\Rightarrow \frac{V_1}{RC} \tau \left(1 - e^{-\infty/\tau}\right) = V_0$$

$$\Rightarrow \frac{V_1}{RC} \tau \cdot 1 = V_0 \Rightarrow V_1 = \frac{V_0 RC}{\tau}$$

$$\boxed{V_1 = \frac{V_0 (C + C_1)}{C_1}}$$



d. The energy dissipated in part C is given by $E = \int_0^\infty i(t)^2 R \, dt = \int_0^\infty \left(\frac{V_1}{R} e^{-t/\tau}\right)^2$

$$= \int_0^\infty R \frac{V_1^2}{R^2} e^{-2t/\tau} = R \frac{V_1^2}{R^2} \frac{\tau}{2} = \frac{R V_1^2 \, R C_{eq}}{2R^2}$$

$$= \frac{V_1^2 C_{eq}}{2} = \frac{1}{2}\left(\frac{V_0 C}{C_{eq}}\right)^2 C_{eq} = \frac{1}{2} \frac{V_0^2 C^2}{C_{eq}}$$

The energy dissipated in pt A is $\frac{CV_0^2}{2}$ from above. Thus circuit A

dissipates more power by a factor of

$\frac{C}{C_{eq}} = \frac{C (C_1 + C)}{C \cdot C_1} = \frac{C_1 + C}{C_1}$, which is clearly greater than 1

$\boxed{\text{So circuit A is more efficient.}}$

e. At $t=0$, $i(t)=0$

At $t=\infty$, inductor is short and $i(\infty) = I_0$

$$\boxed{\begin{aligned}\tau &= L/R\\ i(t) &= I_0(1-e^{-tR/L})\\ v(t) &= I_0 R\, e^{-tR/L}\end{aligned}}$$

The total energy stored in the inductor is $\frac{1}{2} L \cdot i(\infty)^2 = \boxed{\frac{1}{2} L I_0^2}$

The energy dissipated in the resistor is

$$\int_0^\infty \frac{v(t)^2}{R} = \int_0^\infty \frac{1}{R} I_0^2 R^2 e^{-2tR/L}$$

$$= I_0^2 R \cdot \left(0 + \frac{L}{2R}\right) = \boxed{\frac{1}{2} I_0^2 L}$$

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The total energy provided by the source is the sum of these. $\boxed{L I_0^2}$

3a. i From 0 to T, the particular solution $v_p = 1/T$. The homogeneous soln is $v_h = c\, e^{-t/RC}$

Choose c such that $v(0)=0$

$0 = 1/T + c\,e^{0} \Rightarrow c = -1/T$

$v(t) = 1/T - 1/T\, e^{-t/RC} = 1/T\,(1-e^{-t/RC})$

ii. $v(T^-) = 1/T\,(1-e^{-T/RC})$

Thus $v(t) = 1/T\,(1-e^{-T/RC})\, e^{-\frac{(t-T)}{RC}}$

for $t > T$

a iii In the limit as $T \to 0$

$$V(t) = \lim_{T\to 0} \frac{1 - e^{-T/\tau}}{T} e^{-(t-T)/RC}$$

$$= \lim_{T\to\infty} \frac{\frac{1}{RC} e^{-T/RC}}{1} e^{-(t-T)/RC}$$

$$= \frac{1}{\tau} e^{-t/RC} = \boxed{\frac{1}{RC} e^{-t/RC}}$$

b. $\int_{0^-}^{0^+} RC \frac{dv}{dt} + \int_{0^-}^{0^+} v = \int_{0^-}^{0^+} \delta(t)$

$\Rightarrow RC\, v(0^+) + v(0^-) + 0 = 1$

$v(0^+) = 1/RC$

Using the homogeneous soln found in pt A, and choose an appropriate constant

$$\boxed{v(t) = \frac{1}{RC} e^{-t/RC}}$$

c. $i_c = \frac{\delta(t)}{R}$

$\int_{0^-}^{0^+} i_c(t) = \int_{0^-}^{0^+} \frac{\delta(t)}{R} = \frac{1}{R}$

$v(0^+) = \frac{1}{C} \int_{0^-}^{0^+} i_c(t) = \frac{1}{RC}$

$$\boxed{v(t) = \frac{1}{RC} e^{-t/RC}}$$

d. The step response is the solution to

$RC \frac{dv}{dt} + v = u(t)$

This has a particular solution $v(t) = u(t)$

To make $v(0^-) = v(0^+)$, choose the constant in the homogeneous soln appropriately

$v(t) = 1 - 1e^{-t/RC}$

The impulse response is the derivative!

$h(t) = -1 \cdot -\frac{1}{RC} e^{-t/RC} = \boxed{\frac{1}{RC} e^{-t/RC}}$

4. Assuming that the charge Q is applied directly to the capacitor

A. $v(0^-) = V_0$

$v(0^+) = v(0^-) + \frac{Q}{C} = V_0 + \frac{Q}{C}$

After the impulse is delivered, the current source is an open circuit. Thus the voltage across the capacitor is dissipated over the resistor w/ time constant RC

$$\boxed{v(t) = \left(V_0 + \frac{Q}{C}\right) e^{-t/RC}}$$

b. $v(T) = \left(V_0 + \frac{Q}{C}\right) e^{-T/RC}$

Setting this equal to V_0,

$V_0 = V_0 e^{-T/RC} + \frac{Q}{C} e^{-T/RC}$

$V_0 (1 - e^{-T/RC}) = \frac{Q}{C} e^{-T/RC}$

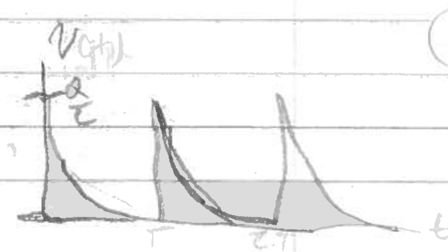
$$V_0 = \frac{Q e^{-T/RC}}{C (1 - e^{-T/RC})}$$

For $T \gg RC$,

$V_0 \approx \frac{Q \cdot 0}{C} = 0$

$v(t) = \frac{Q}{C} e^{-t}$

$V_0 = 0$



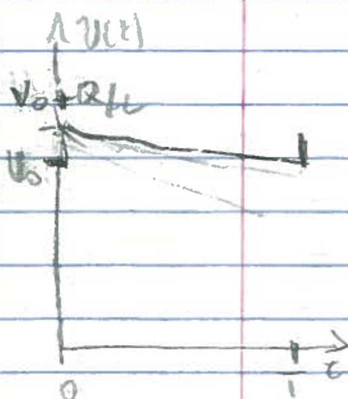
Consider $RC \gg T$. Then

$v(t) = (V_0 + Q/C)e^{-T/RC} = (V_0 + Q/C)e^{0}$

$= V_0 + Q/C \Rightarrow v(t)$ is constant.

But we are equating $v(T^-) = V_0$. Thus

$v(T^-) = V_0 + Q/C = V_0$.



The only way this can be true is if $Q/C = 0 \Rightarrow Q = 0$, i.e. there is no impulse. In reality, the situation is not quite so extreme because RC is not infinitely greater than T. In this case, $v(t)$ decays slightly from $t=0$ to $t=T$, and $V_0 \gg Q/C$

Since the voltage returns to V_0 at every period, the response to this input is also periodic. v^* repeats at every period.

$v = v^*(t) + v_1(t)$

where $\boxed{v_1(t) = \sum_{n=1}^{\infty} v^*(t - nT)}$