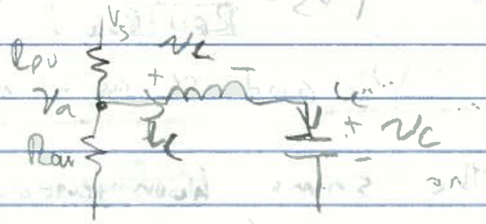
JC $\frac{10}{10}$

Dan Ports
6.002 PS9

(3) For $0 \leq t < T$,



$v_C(0^-) = V_s$

$i_L(0^-) = 0$

$v_O = v_C$

Applying nodal analysis at v_A, we find

$$\frac{(v_A - V_s)}{R_{pu}} + \frac{v_A}{R_{on}} + i_L = 0 \Rightarrow v_A\left(\frac{1}{R_{pu}} + \frac{1}{R_{on}}\right) = -i_L + \frac{V_s}{R_{pu}}$$

$$\Rightarrow v_A = \left(i_L + \frac{V_s}{R_{pu}}\right)(R_{pu} \| R_{on})$$

Let $R = R_{pu} \| R_{on}$

By KCL, $i_L = i_C$; By KVL, $v_A = v_L + v_C \Rightarrow v_C = v_A - v_L$

From element laws, $i_L = i_C = C\frac{dv_C}{dt}$

and $v_L = L\frac{di_L}{dt} = LC\frac{d^2v_C}{dt^2}$

$$v_C = \left[-C\frac{dv_C}{dt} + \frac{V_s}{R_{pu}}\right]R - LC\frac{d^2v_C}{dt^2}$$

$$\Rightarrow \ddot{v}_C + \frac{RC}{LC}\dot{v}_C + \frac{v_C}{LC} = \frac{V_s R}{R_{pu}LC}$$

By inspection, the particular soln is $v_{Cp} = \frac{V_s R}{R_{pu}}$

$$D(s) = s^2 + \frac{R}{L}s + \frac{1}{LC} = 0$$

$$\Rightarrow s = \frac{-R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

We know that $\sqrt{\frac{L}{C}} > R/2 \Rightarrow \frac{L}{C} > R^2/4 \Rightarrow \frac{1}{LC} > \frac{R^2}{4L^2}$

So $s = \frac{-R}{2L} \pm j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = -\alpha \pm \omega j$

$\boxed{\left(\text{Let } \omega = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \text{ and } \alpha = \frac{R}{2L}\right)}$

The homogeneous response

$$v_{Ch} = e^{-R/2L}(a\cos\omega t + b\sin\omega t)$$

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$\therefore\; v_C(t) = v_{Cp} + v_{Ch} = \frac{V_S R}{R_{PU}} + e^{-\alpha t}\left(a\cos(\omega_d t) + b\sin(\omega_d t)\right)$

The initial conditions $v_C(0) = V_S$ and $i_C(0) = 0$ imply that $a = V_S - V_S R / R_{PU}$, $b = 0$

$$\boxed{v_O(t) = v_C(t) = \frac{V_S R_{ON}}{R_{PU}+R_{ON}} + e^{-\alpha t}\left(V_S - \frac{V_S R_{ON}}{R_{PU}+R_{ON}}\right)\cos(\omega_d t)}$$

for $0 < t < T$; ω_d and α as above, $R = R_{PU} \| R_{ON}$

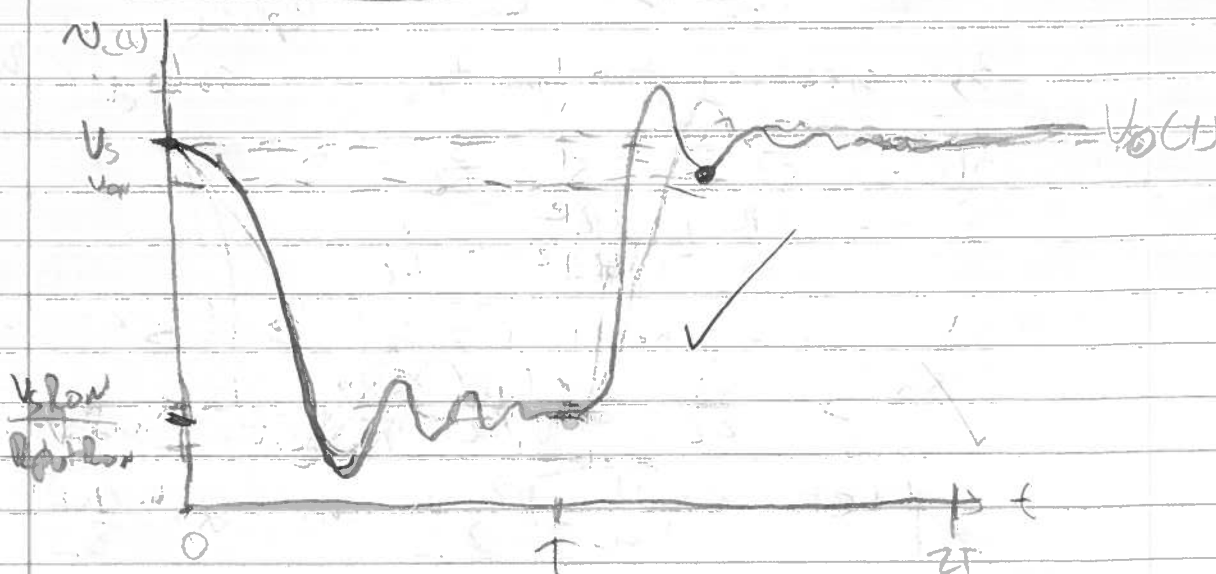
For $T < t < 2T$, the same homogeneous solution applies, except that no current flows through the MOSFET ($R_{ON} \to \infty$ and $R_{PU} \| R_{ON} = R_{PU}$), and the particular solution and initial conditions are different.

The new particular solution is $v_O(t) = V_S$, and the initial conditions are $v_O(T) = V_S \frac{R_{ON}}{R_{PU}+R_{ON}}$ and $\dot{v}_O(T) = \frac{1}{C} i_C(T) = 0$.

The solution is:

$$\boxed{v_O(t) = V_S + \left(\frac{V_S R_{ON}}{R_{PU}+R_{ON}} - V_S\right) e^{-\alpha(t-T)} \cos(\omega_d (t-T))}$$

for $T < t < 2T$; ω_d and α as above except $R = R_{PU}$



b. The minimum of $V_o(t)$ after the output switches shifts high must be greater than V_{OH}. This point exists when $\omega_d(t-T)=2\pi \Rightarrow$ $t = T + \frac{2\pi}{\omega_d}$. This point is indicated on the graph. Assuming $R_{on} \ll R_{pu}$ and the indicated values:

$$v_C(t) = V_S\left(1-e^{-\alpha(t-T)}\cos(\omega_d(t-T))\right)$$

At the minimum point,

$$V_{OH} = 0.8V_S < V_S\left(1-e^{-\alpha 2\pi/\omega_d}\cdot 1\right)$$

$$\Rightarrow 0.8 < 1-e^{-2\pi\alpha/\omega_d} \Rightarrow e^{-2\pi\alpha/\omega_d} > 0.2$$

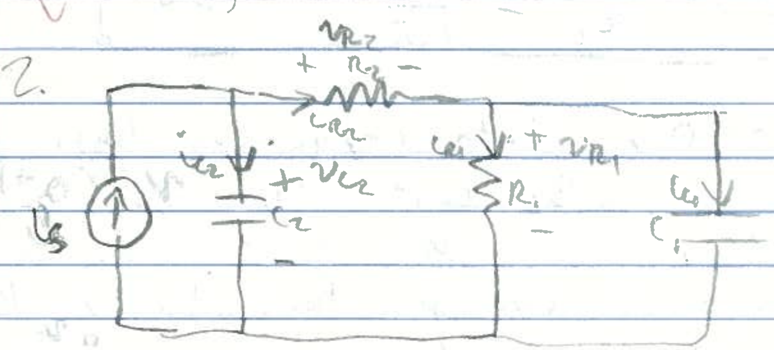
$$\Rightarrow -\frac{2\pi\alpha}{\omega_d} > \ln(0.2) \Rightarrow \frac{2\pi R_{pu}}{2L\sqrt{1/LC - \frac{R_{pu}^2}{4L^2}}} < -\ln(0.2)$$

$$\Rightarrow \frac{2\pi \cdot 1k\Omega}{2L\sqrt{1/L\cdot 10^{-2}pF - \frac{(1k\Omega)^2}{4L^2}}} < 1.609 \Rightarrow \frac{1}{\sqrt{L/10^{-2}pF - \frac{(1k\Omega)^2}{4}}} < 0.000512S$$

$$\Rightarrow \frac{L}{10^{-2}pF} - (250.2)^2 < 3.810\times10^6\ \Omega^2$$ close.

$$\Rightarrow \frac{L}{10^{-2}pF} < 3.873\times10^6\ \Omega^2 \Rightarrow \boxed{L < 3.873\times10^{-8}\ H}$$

2 2.



By KCL, $i_S = i_{R_2} + i_{C_2}$; $i_{R_2} = i_{R_1} + i_{C_1}$

$$i_S = i_{R_1} + i_{C_1} + i_{C_2}$$

$$i_S = \frac{v_{C_1}}{R_1} + C_1\dot{v}_{C_1} + i_{C_2}$$

$$v_{C_2} = v_{R_2} + v_{R_1} = i_{R_2}R_2 + v_{C_1} = R_2(i_{R_1}+i_{C_1}) + v_{C_1}$$
$$= R_2\frac{v_{C_1}}{R_1} + R_2C_1\dot{v}_{C_1} + v_{C_1}$$

$$i_{C_2} = C_2 \dot{v}_{C_2} = C_2\left(\frac{R_2}{R_1}\dot{v}_{C_1} + R_2 C_1 \ddot{v}_{C_1} + \dot{v}_{C_1}\right)$$

$$I_S = \frac{v_{C_1}}{R_1} + C_1 \dot{v}_{C_1} + \frac{C_2 R_2}{R_1}\dot{v}_{C_1} + C_1 C_2 R_2 \ddot{v}_{C_1} + C_2 \dot{v}_{C_1}$$

$$C_1 C_2 R_2 \ddot{v}_{C_1} + \left[C_1 + C_2 R_2/R_1 + C_2\right]\dot{v}_{C_1} + \frac{1}{R_1} v_{C_1} = I_S$$

$$\boxed{\ddot{v}_{C_1} + \left[\frac{1}{C_2 R_2} + \frac{1}{C_1 R_1} + \frac{1}{C_1 R_2}\right]\dot{v}_{C_1} + \frac{1}{C_1 C_2 R_1 R_2} v_{C_1} = \frac{I_S}{C_1 C_2 R_2}}$$

b. i. $p(x) = x^2 + \left(\frac{1}{C_2 R_2} + \frac{1}{C_1 R_1} + \frac{1}{C_1 R_2}\right)x + \frac{1}{C_1 C_2 R_1 R_2} = 0$

$\Rightarrow x^2 + \left(2\,Hz + \frac{3}{2}\,Hz + \frac{1}{2}\,Hz\right)x + 3/s^2 = 0$

$\Rightarrow x^2 + 4\,Hz\, x + 3/s^2 = 0$

$\Rightarrow \boxed{x = -1\,Hz \text{ or } x = -3}$

ii. $\boxed{v_{C_{1h}}(t) = A_1 e^{-t} + A_2 e^{-3t}}$

iii. By inspection from the equation,

$\boxed{v_{C_{1p}}(t) = \frac{I_S}{R_1} = \frac{I_S}{1\Omega} \quad t > 0}$

iv. At $t = 0^-$, both capacitors have zero voltage. By continuity, $\boxed{v_{C_1}(0^+) = 0}$

Similarly, since v_{C_2} is continuous, $v_{C_2}(0^+) = 0$.

This means no current flows through R_1 or R_2, so no current can flow through C_1. $\boxed{\left.\frac{dv_{C_1}}{dt}\right|_{t=0^+} = 0}$

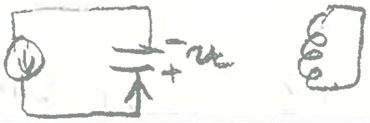
v. $v_{C_1}(t) = I_S/R_1 + A_1 e^{-t} + A_2 e^{-3t}$

$v_{C_1}(0) = 0 \Rightarrow A_1 + A_2 = -I_S/R_1$

$\dot{v}_{C_1}(0) \Rightarrow -1A_1 - 3A_2 = 0$

$\Rightarrow A_1 = -3/2\, I_S/1\Omega, \quad A_2 = 1/2\, I_S/1\Omega$

$$\boxed{v_{C_1}(t) = \frac{I_S}{1\Omega} - \frac{3 I_S}{2\Omega} e^{-t} + \frac{I_S}{2\Omega} e^{-3t}}$$



3. A.

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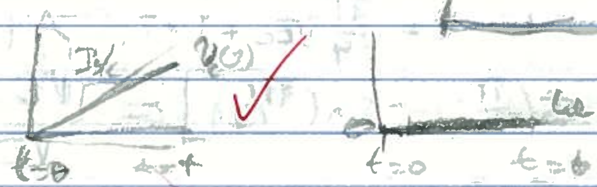
Before time 0, there is no energy, voltage, or current. For $t>0$, the inductor is short-circuited so no current flows through it. The current I_s is applied to the capacitor.

$$C\frac{d}{dt}v_c = I_s \Rightarrow \boxed{v_c(t) = \frac{I_s}{C}t, \quad i_L(t) = 0}$$

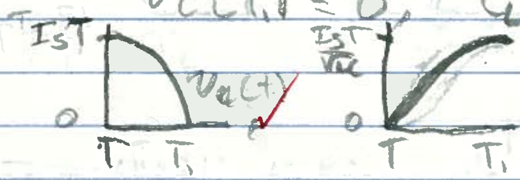


b. At this point the circuit is an undriven LC circuit. The initial $V(T) = I_sT/C$, $I(T) = 0$. The differential equation for the circuit is $\frac{d^2v(t)}{dt^2} + \frac{1}{LC}v(t) = 0$. It is solved by

$$\boxed{\begin{aligned} v_c(t) &= \frac{I_sT}{C}\cos\left(\frac{t-T}{\sqrt{LC}}\right) \\ i_L(t) &= -C\frac{dv_c}{dt} = \frac{I_sT}{\sqrt{LC}}\sin\left(\frac{t-T}{\sqrt{LC}}\right) \end{aligned}}$$

When the cosine term is zero, the sine term is 1

$v_c(T_1) = 0$, $i_L(T_1) = I_sT/\sqrt{LC}$



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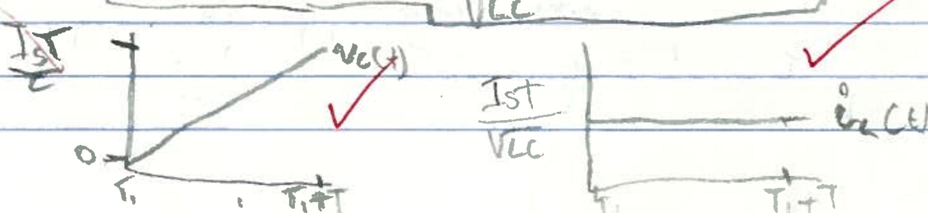
c. The current through the inductor does not change because the voltage across the inductor is zero. The current source charges the inductor exactly as in part a.

$$C\frac{dv_c}{dt} = I_s \Rightarrow \boxed{v_c(t) = \frac{I_s}{C}(t-T_1)}$$

$$\boxed{i_L(t) = i_L(T_1) = \frac{I_sT}{\sqrt{LC}}}$$



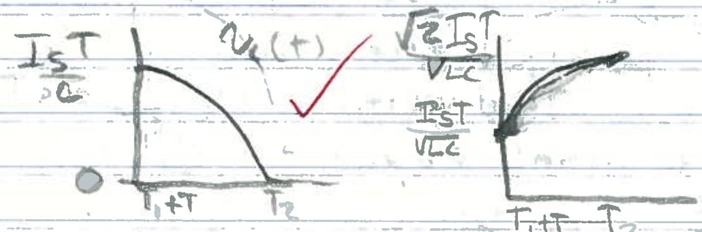
d. This is an undriven LC circuit with sinusoidal response.

$\frac{d^2 v_c}{dt^2} + \frac{1}{LC} v_c = 0$ The initial conditions imply that $\boxed{v_c(t) = \frac{I_s T}{C}\cos\left(\frac{t-(T_1+T)}{\sqrt{LC}}\right) - \frac{I_s T}{C}\sin\left(\frac{t-(T_1+T)}{\sqrt{LC}}\right)}$

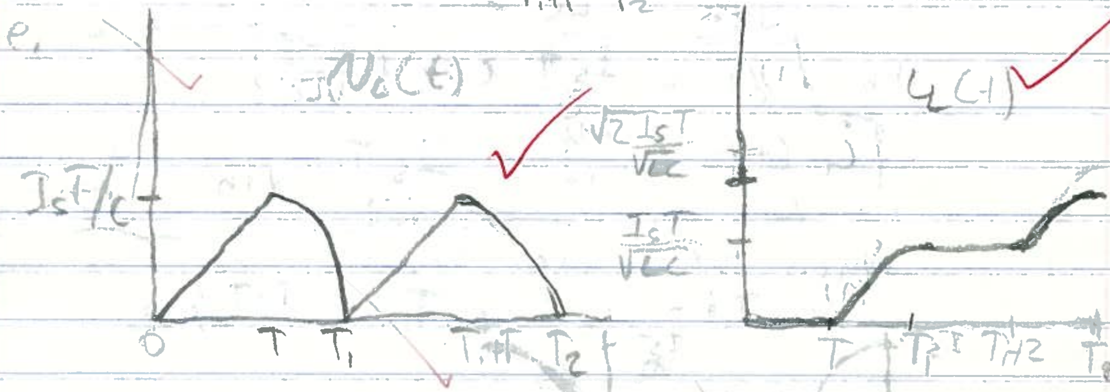
$$\boxed{\Rightarrow i_L(t) = -C\dot{v}_c = \frac{I_s T}{\sqrt{LC}}\cos\left(\frac{t-(T_1+T)}{\sqrt{LC}}\right) + \frac{I_s T}{\sqrt{LC}}\sin\left(\frac{t-(T_1+T)}{\sqrt{LC}}\right)}$$

$v_c(t) = 0$ at $t = \frac{\pi}{4}\sqrt{LC} + T_1 + T$. At this time

$i_L(T_2) = \frac{I_s T}{\sqrt{LC}}\left[\cos\left(\frac{\pi}{4}\right) - \sin\left(\pi/4\right)\right] = \frac{\sqrt{2} I_s T}{\sqrt{LC}}$



e.



2 9.4. a. $v_c(t) = I_sT/C$ from above

$\Rightarrow$ energy in cap $= \frac{1}{2} C V_0^2 = \boxed{\frac{1}{2} I_s^2 T^2 / C}$

b. Energy in inductor $= \frac{1}{2} L i_L^2$. The energy

$\frac{1}{2} I_s^2 T^2 / C = \frac{1}{2} L\, i_L(T_1)^2 \Rightarrow i_L(T_1)^2 = \frac{I_s^2 T^2}{LC}$

$\Rightarrow \boxed{i_L(T_1) = \frac{I_s T}{\sqrt{LC}}}$

c. The capacitor voltage at time T_1+T is I_sT/C from above. again

$E = \frac{1}{2} C v_c(T_1+T)^2 = \boxed{\frac{1}{2} \frac{I_s^2 T^2}{C}}$

d. The initial energy in the inductor is

$$\tfrac{1}{2} L i_L(T_1+T)^2 = \tfrac{1}{2} L \frac{I_s^2 T^2}{LC} = \tfrac{1}{2} \frac{I_s^2 T^2}{C}$$

The capacitor contains energy $\tfrac{1}{2} I_s^2 T^2 / C$ which is transferred to the inductor, giving it a final energy of $I_s^2 T^2 / C$

$$\tfrac{1}{2} L i_L(T_2)^2 = I_s^2 T^2 / C$$

$$i_L(T_2)^2 = \frac{2 I_s^2 T^2}{LC}$$

$$\boxed{i_L(T_2) = \frac{\sqrt{2}\, I_s T}{\sqrt{LC}}}$$ ✓, as expected.

e. After n cycles, the energy on the inductor will be $n \cdot \tfrac{1}{2} I_s^2 T^2 / C$

$$\tfrac{1}{2} L \cdot i_L^2 = \tfrac{1}{2} n I_s^2 T^2 / C$$

$$\Rightarrow i_L^2 = \frac{n I_s^2 T_i^2}{LC}$$ ✓

$$\Rightarrow \boxed{i_L = \frac{\sqrt{n}\, I_s T}{\sqrt{LC}}}$$