Dan Ports
6.003 PS 10

3/3

1. 11.49 a. DC gain $= H(0) = \frac{6a}{a} = \boxed{6}$

b. The impulse response is $\mathcal{L}^{-1}\left(\frac{6a}{s+a}\right)$
$= 6a\, e^{-at} u(t)$. So the time constant $\boxed{\tau = 1/a}$

c. $|H(\omega)| = \frac{1}{\sqrt{2}} |H(0)| = \frac{1}{\sqrt{2}} 6$

$$\frac{6a}{\sqrt{a^2+\omega^2}} = \frac{6}{\sqrt{2}}$$

$$6^2 a^2 = \frac{6^2}{2}(a^2+\omega^2)$$

$$6^2 a^2 (1 - 1/2) = 6^2/2 \cdot \omega^2$$

$$6^2 a^2/2 = \frac{6^2}{2}\omega^2$$

$$\omega^2 = a^2 \Rightarrow \omega = a$$

The bandwidth is $\boxed{a}$.

d. By Black's formula,

$$\frac{Y(s)}{X(s)} = \frac{\frac{6a}{s+a}}{1 + k\frac{6a}{s+a}} = \frac{1}{\frac{s+a}{6a} + k} = \frac{6a}{s+a+k6a}$$

The DC gain is $\frac{6a}{k6a+a} = \boxed{\frac{1}{1/6 + k}}$

The impulse response is $\mathcal{L}^{-1}\left(\frac{6a}{s+(a+k6a)}\right)$
$= 6a\, e^{-(a+k6a)t} u(t)$.
So the time constant $\boxed{\tau = \frac{1}{a+k6a}}$

To find the bandwidth, we find ω such that

$$|H(j\omega)| = \frac{1}{\sqrt{2}}|H(0)| = \frac{1}{\sqrt{2}}\frac{Ga}{KGa+a}$$

$$\frac{Ga}{\sqrt{\omega^2+(a+KGa)^2}} = \frac{1}{\sqrt{2}}\frac{Ga}{KGa+a}$$

$$\frac{G^2a^2}{\omega^2+(a+KGa)^2} = \frac{1}{2}\frac{G^2a^2}{(KGa+a)^2}$$

$$(KGa+a)^2 = \frac{1}{2}\left[\omega^2+(KGa+a)^2\right]$$

$$\frac{1}{2}(KGa+a)^2 = \frac{1}{2}\omega^2$$

$$\omega = (KGa+a).$$

So the bandwidth is $\boxed{KGa+a}$

c. We want the bandwidth $KGa+a$ to equal $2a$, so we choose $\boxed{K=1/G}$.

Then the time constant $\boxed{\tau = \frac{1}{a+KGa} = \frac{1}{2a}}$

and the DC gain is $\frac{1}{1/G+K} = \frac{1}{2/G} = \boxed{\frac{G}{2}}$

2. a. We apply Black's formula.

$$H(s) = \frac{\frac{k}{(s-1)^2}}{1 + \frac{k(s+1)}{(s-1)^2}} = \boxed{\frac{k}{(s-1)^2 + k(s+1)}}$$

✓

b. $$H(s) = \frac{k}{(s-1)^2 + k(s+1)} = \frac{k}{s^2 + (k-2)s + (k+1)}$$

Applying the Routh-Hurwitz criterion, we must satisfy $k-2>0$

and $k+1>0$ ✓

for stability so we require $\boxed{k>2}$

3. 11.45. a. Let $C(s) = \frac{s-2}{s+3}$. Then the overall system fcn $= \frac{1}{(s+1)(s-2)} \cdot \frac{(s-2)}{(s+3)} = \frac{1}{(s+1)(s+3)}$

Which is stable since both poles are in the left half-plane. But this is not useful in practice because the pole of H and zero of C must be at exactly the same location for cancellation. If they are off even slightly, this does not work.

b. Let $C(s) = k$. Then the overall sys fcn is

$$\frac{\frac{k}{(s+1)(s-2)}}{1 - \frac{k}{(s+1)(s-2)}} = \frac{k}{(s+1)(s-2) - k}$$

$$= \frac{k}{s^2 - s - 2 + k}$$

The s coefficient in the denominator is always negative, so by Routh-Hurwitz the system cannot be stabilized.

c. Let $C(s) = K(s+a)$. Then the sys fcn is

$$\frac{K(s+a)}{(s+1)(s-2) + K(s+a)} = \frac{K(s+a)}{s^2 + (K-1)s + (Ka-2)}.$$

So by Routh-Hurwitz the system is stable if $\boxed{K > 1 \quad \text{and} \quad K > 2/a}$

d). In this case the sys tfn is

$$\frac{K(s+2)}{s^2 - s - 2 + Ks + 2K}.$$ We want the

denominator to have the form $s^2 + \omega_n s + \omega_n^2$

So

$$s^2 + (K-1)s + (2K-2) = s^2 + \omega_n s + \omega_n^2$$

$$\Rightarrow (K-1)^2 = 2K-2$$

$$\Rightarrow K^2 - 3K + 3 = 0 \Rightarrow \boxed{K = 3}$$ ✓

e) i). Let $C(s) = K\left(\frac{s+1/2}{s+2}\right)$.

Then the sys tfn is

$$\frac{K(s+1/2)/(s+2) \quad 1/(s+1)(s-2)}{1 + \frac{K(s+1/2)}{(s+2)} \frac{1}{(s+1)(s-2)}}$$

$$= \frac{K(s+1/2)}{s^3 - 4s + s^2 - 4 + Ks + 1/2 K}$$

$$= \frac{K(s+1/2)}{s^3 + s^2 + (K-4)s + (K/2 - 4)}$$

So we can stabilize the system if

we have $K-4 > 0$, $K/2 - 4 > 0$, $K-4 > K/2 - 4$

$$\Rightarrow \boxed{K > 8}$$ ✓

ii) Let $C(s) = k\left(\frac{s+3}{s+2}\right)$.

Then the sys fcn is

$$\frac{k\left(\frac{s+3}{s+2}\right)\frac{1}{(s+1)(s-2)}}{1 + k\frac{(s+3)}{(s+2)}\frac{1}{(s+1)(s-2)}}$$

$$= \frac{k(s+3)}{(s+1)(s-2)(s+2) + k(s+3)}$$

$$= \frac{k(s+3)}{s^3 + s^2 - 4s - 4 + ks + 3k}$$

$$= \frac{k(s+3)}{s^3 + s^2 + (k-4)s + (3k-4)}$$

By Routh-Hurwitz, for stability we must have $k-4>0$, $3k-4>0$, and $k-4 > 3k-4$.

The first two conditions require $k>4$. But then $k-4 < 3k-4$, meaning the third condition cannot be met. So we cannot stabilize the system this way.

4. The transfer function is

$$H(s) = \frac{K_u(s)}{(s+1)(s+10) + K_u(s)} = \frac{K_u(s)}{s^2 + 11s + 10 + K_u(s)}.$$

a. The step response of this system reaches a steady-state of $0.75 \neq 1$. So this must be the proportional controller $K_u(s) = K$. Applying the final value theorem, we have

$$\lim_{s \to 0} s \frac{1}{s} \frac{K}{s^2 + 11s + 10 + K} = \lim_{t \to \infty} s(t) = .75$$

So $\frac{K}{10+K} = .75 \Rightarrow \boxed{K = 30}$

b. The integral controller K_2 gives

$$H(s) = \frac{K/s}{s^2 + 11s + 10 + K/s} = \frac{K}{s^3 + 11s^2 + 10s + K}$$

So we must have $K > 0$ and $11 \cdot 10 > K$ for stability by Routh-Hurwitz. Thus we need $\boxed{0 < K < 110.}$

c. As k becomes large, the root locus branches approach the open-loop zeros. We can see from the plot that these zeros are at $-\infty$ and approximately $-1.5 \pm 3i$.

So we want the open-loop transfer function

$$\frac{k(1 + a/s + b/s)}{(s+1)(s+10)}$$ to have

zeros at $-1.5 \pm 3i$.

Its zeros are at

$$k(1 + a/s + b/s) = 0$$

$$\Rightarrow s + a + bs^2 = 0$$

$$\Rightarrow s = \frac{-1 \pm \sqrt{1-4ab}}{2b}$$

We want the real part, $-1/2b$ to be -1, so $\boxed{b = 1/2}$ ✓

Then we want $\sqrt{1-4ab} = \sqrt{1-2a} = 3i$

$$\Rightarrow 1-2a = -9 \Rightarrow -2a = -10$$

$$\Rightarrow \boxed{a = 10}$$

this is right...

then $a = 5$, not 10

3 a. $W(s) = P(s)\,100/s$

Matlab plot attached. The surface temperature reaches a steady-state value of 80°, while the header temperature is 100°. This difference in temperature could be explained by heat loss at the surface causing a cooling effect.

b. Using Black's formula,

$$\frac{W(s)}{\lambda(s)} = \frac{KG\beta}{1+KG\beta} = \frac{KP(s)}{1+KP(s)}$$

$$= \frac{1}{1/KP(s)+1}$$

Applying the final value theorem, the steady-state step response magnitude is

$$\lim_{s\to 0} \frac{1}{1/KP(s)+1} = \frac{1}{50/40K+1}$$

We want this to be within 2% of 1, i.e. equal to 0.98.

$$\frac{1}{\frac{50}{40K}+1} = 0.98 \Rightarrow \frac{50}{40K} = 1-\frac{1}{0.98} = 0.0204$$

$$\Rightarrow \boxed{K = 61.25}$$

Matlab plot attached. Plots are also attached for $K = 50$ and

K reduces the steady-state error and the time needed to reach steady state. However, it also increases the amplitude of the transient's peaks. This could potentially kill the fish.

c. The heater temperature z is given by

$$\frac{Z(s)}{X(s)} = \frac{K\,G(s)}{1 + K\,G(s)P(s)}$$ via Black's formula.

A plot of the heater temp's step response w/ $K = 61.25$ as before is attached. The maximum heater temperature is at time zero, which is $100K = 6125$, which is extremely high.

d. Let $G(s) = 1 + 0.05s$, $K = 1$. Then

$$\frac{W(s)}{X(s)} = \frac{K\,G(s)P(s)}{1 + K\,G(s)P(s)} = \frac{1 + 0.05s}{1 + (1+0.05s)P(s)}.$$

This adds a zero at $s = -20$.

As before, the steady-state response to $u(t)$ is $\lim_{s\to 0} s\frac{1}{s}\frac{W(s)}{X(s)}$

$$= \lim_{s\to 0} \frac{K(1+0.05s)P(s)}{1+K(1+0.05s)P(s)} = \frac{K P(0)}{1 + K P(0)} = \frac{K\,50/40}{1 + K\,50/40}.$$

We set this to .98 as before and find $K = 61.25$ as before.

A Matlab plot of $w(t)$ is attached.

e. Now let $G(s) = 1 + 0.05s + 1/s$

Then $H(s) = \frac{W(s)}{X(s)} = \frac{KG(s)P(s)}{1+KG(s)P(s)}$.

The ~~steady-state response to~~ $u(t)$ is, by the FVT,

$$\lim_{s\to 0} s\frac{1}{s}\frac{KG(s)P(s)}{1+KG(s)P(s)} = \lim_{s\to 0}\frac{K(1+0.05s+1/s)P(s)}{1+K(1+0.05s+1/s)P(s)}$$

$$= \lim_{s\to 0}\frac{KP(s)+K/s\,P(s)}{1+KP(0)+K/s\,P(0)} =$$

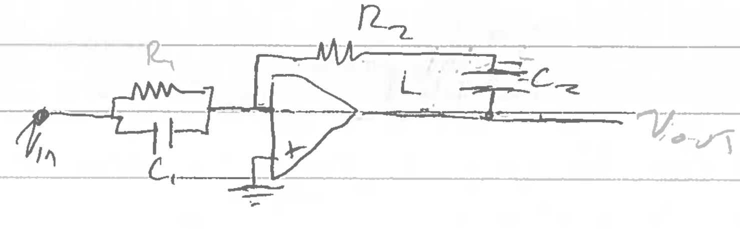
$$\lim_{s\to 0}\frac{sKP(s)+KP(s)}{s+sKP(s)+KP(s)} = \lim_{s\to 0}\frac{KP(s)}{KP(s)} = 1$$

So there is no steady-state error.

A Matlab plot of the response to $x(t) = 100\,u(t)$ with the same value of K as before $(K = 61.25)$.

We see that this has all the desired characteristics.

f. Consider, e.g., the circuit



Then $G(s) = \dfrac{-\,R_2 + 1/C_2 s}{\dfrac{R_1\frac{1}{C_1 s}}{R_1 + \frac{1}{C_1 s}}}$

$$= R_2R_1 + \frac{1}{C_1 s}R_2 + \frac{R_1}{C_2 s} + \frac{1}{C_1C_2 s^2}$$

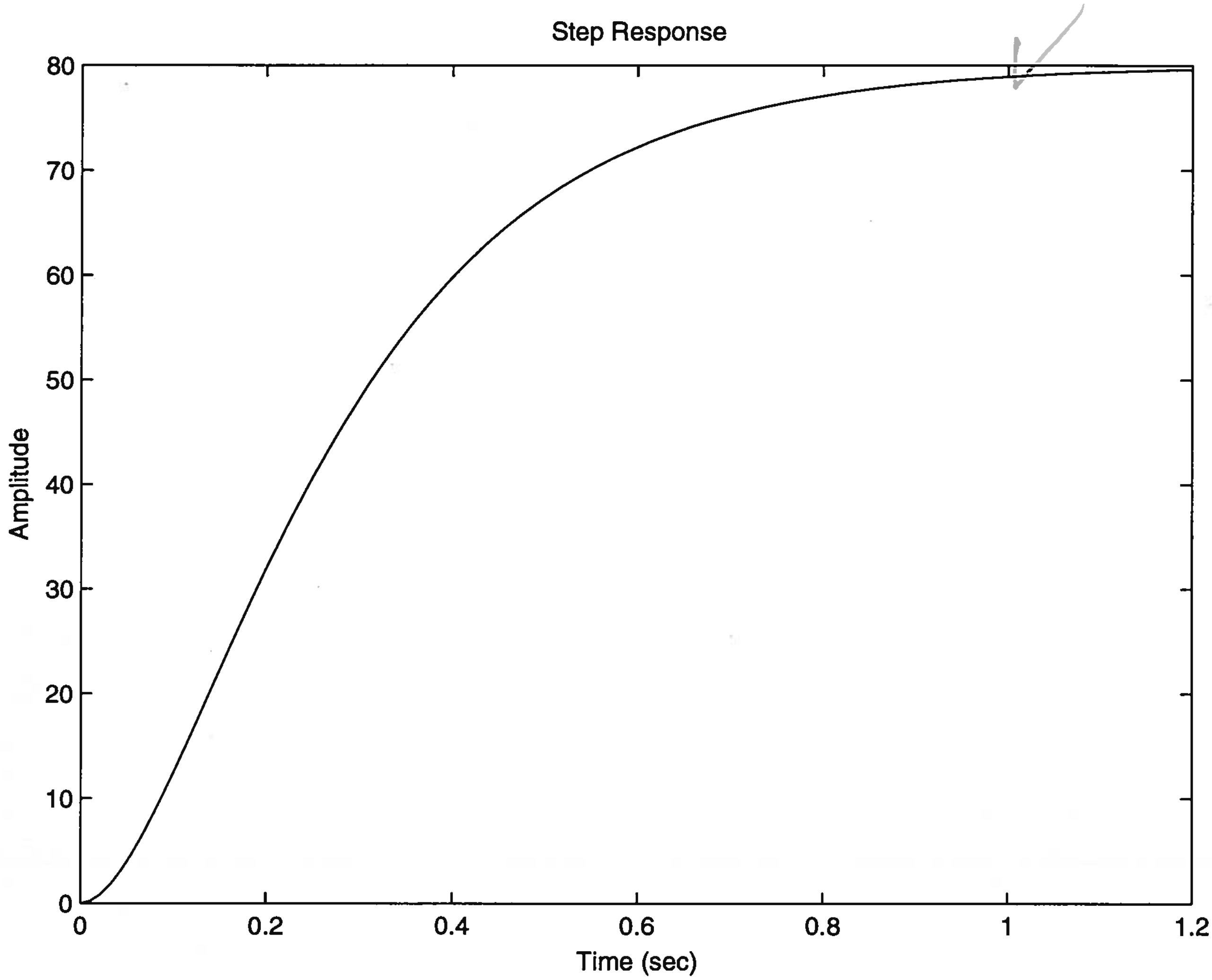
$$= -\left[C_1 R_2 s + \left(\frac{R_2}{R_1} + \frac{1}{C_1 C_2} \right) + \frac{1}{R_1 C_1} \frac{1}{s} \right]$$

We can make this equal the desired $1 + 0.05 s + 1/s$ by setting,

$$\boxed{\begin{aligned} C_1 R_2 &= 0.05 \\ R_2/R_1 + 1/C_1C_2 &= 1 \\ \text{and} \quad \frac{1}{R_1 C_1} &= 1 \end{aligned}}$$

5a

Response of system w/o feedback to 100 u(t)

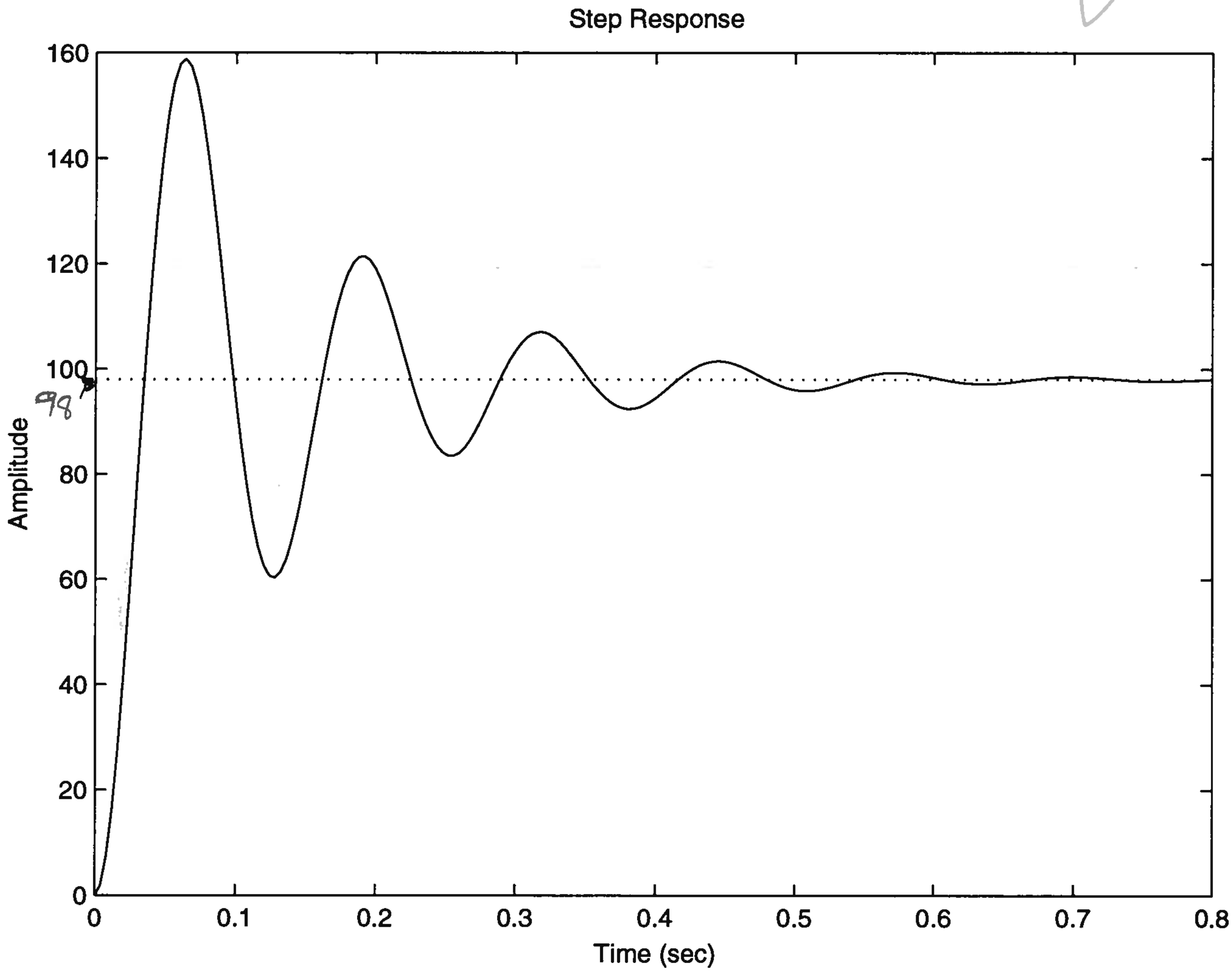


```
diary on
s=tf('s')

Transfer function:
s

P=40/(s^2+15*s+50)

Transfer function:
      40
---------------
s^2 + 15 s + 50

step(100*P)

diary off
```

15b

Response to 100 u(t)

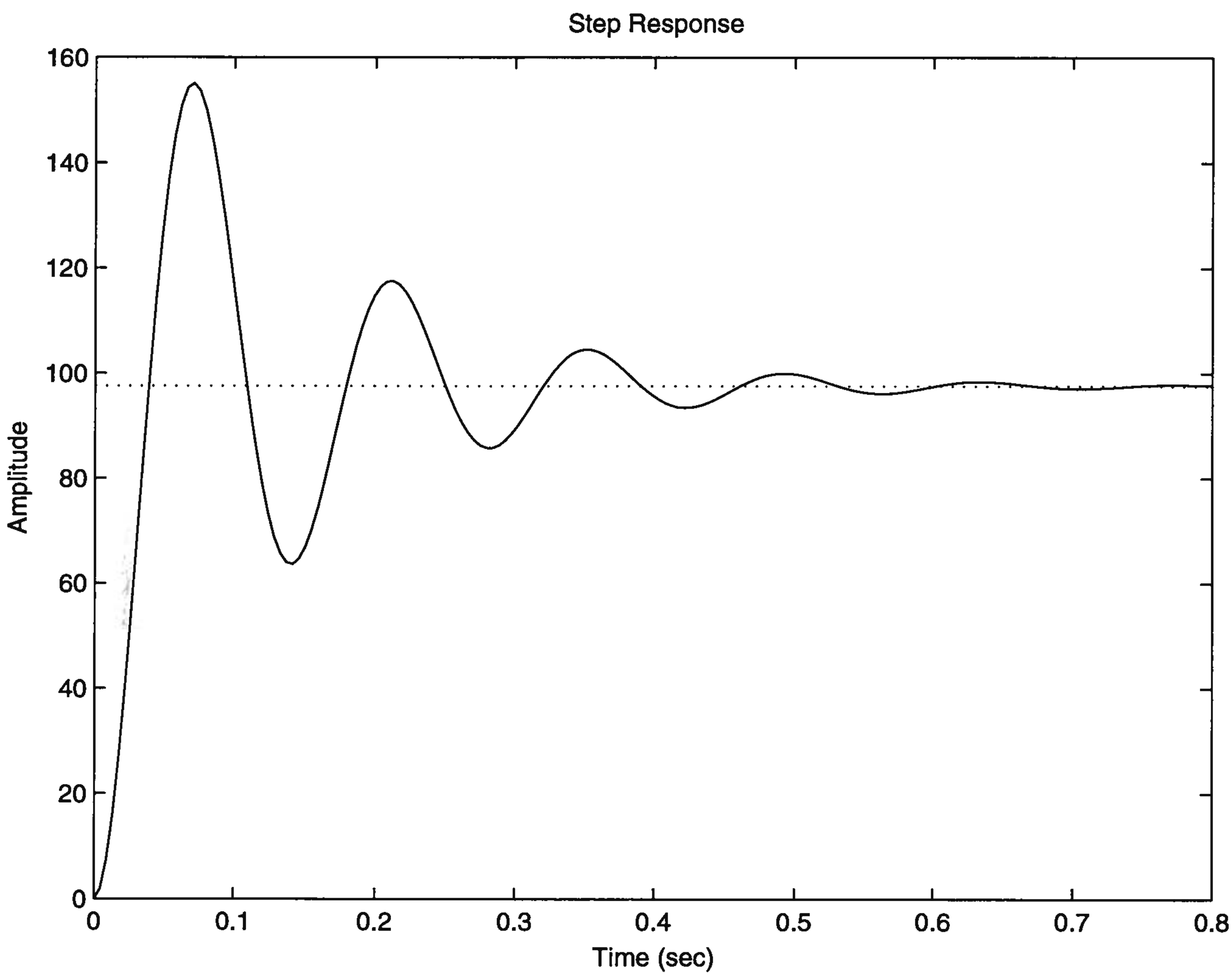
w/ G=1, k= 61.25



5b

Response to 100u(t),

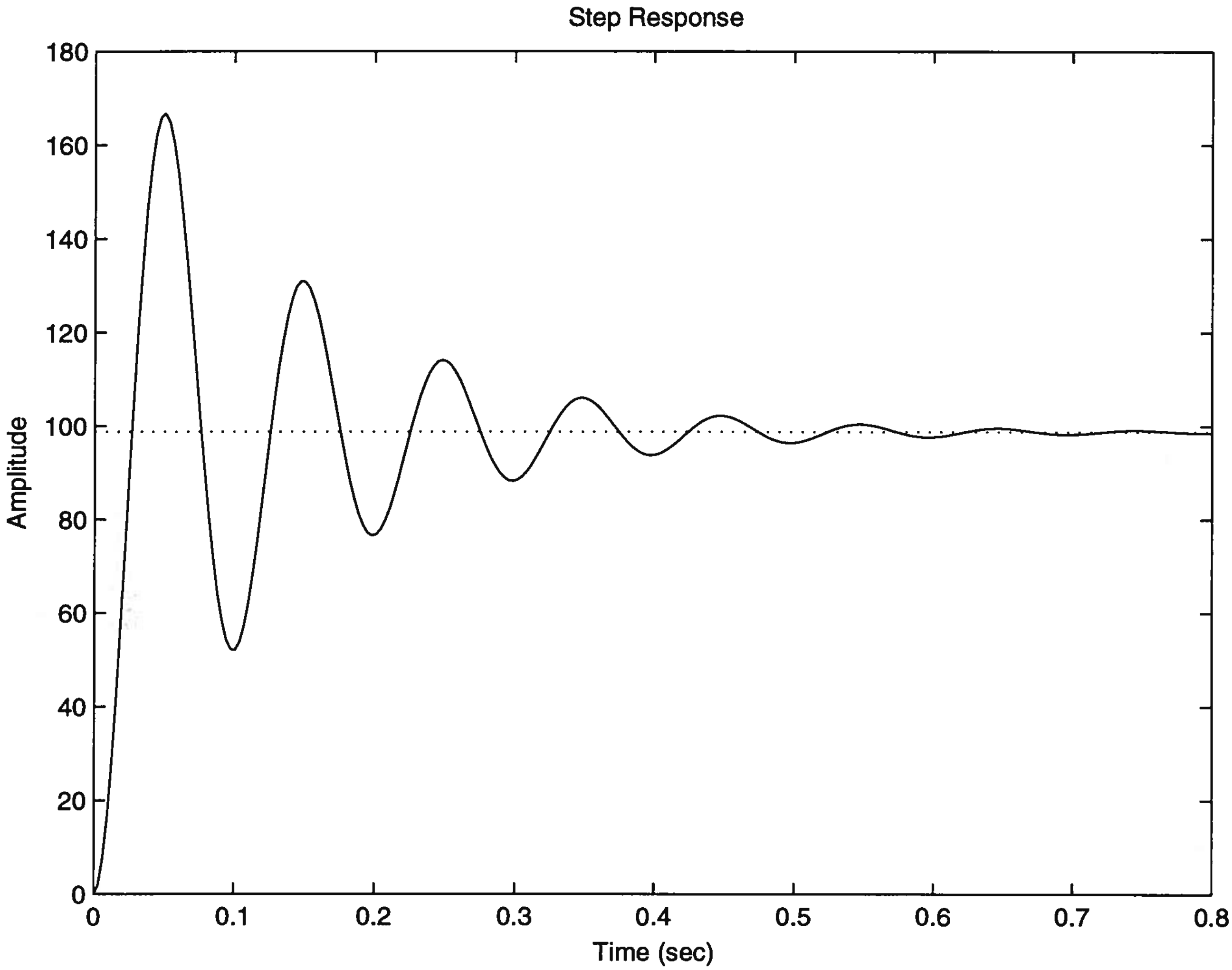
$G=1$, $K=50$



L5b

Response to 100 u(t)

$\theta=1, k=100$

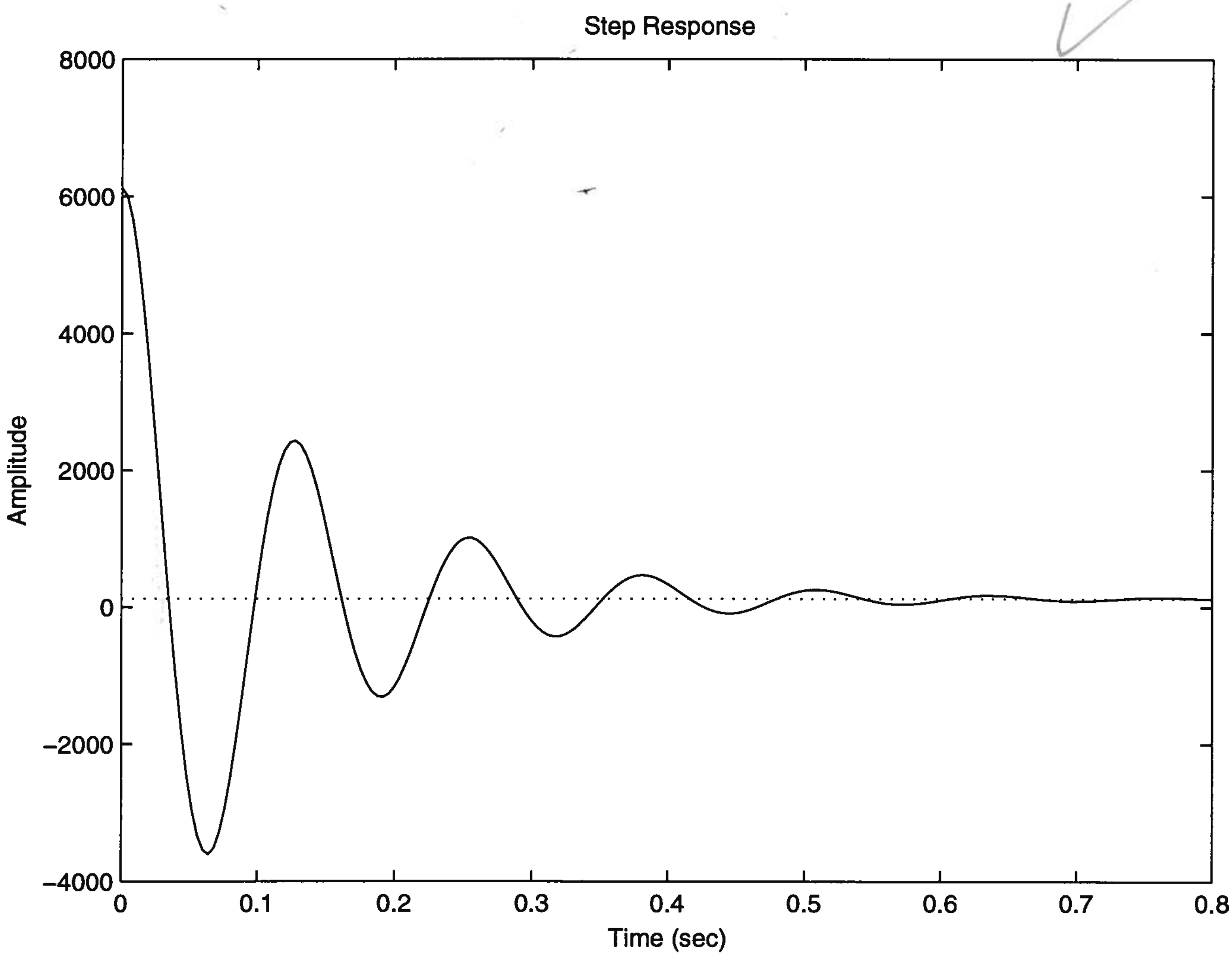


```
diary on
s=tf('s')

Transfer function:
s

P=40/(s^2+15*s+50)

Transfer function:
      40
---------------
s^2 + 15 s + 50

K=61.25

K =

   61.2500

H=(K*P)/(1+K*P)

Transfer function:
       2450 s^2 + 36750 s + 122500
-----------------------------------------
s^4 + 30 s^3 + 2775 s^2 + 38250 s + 125000

step(100*H)

K=50

K =

    50

H=(K*P)/(1+K*P)

Transfer function:
       2000 s^2 + 30000 s + 100000
-----------------------------------------
s^4 + 30 s^3 + 2325 s^2 + 31500 s + 102500

step(100*H)

K=100

K =

   100

H=(K*P)/(1+K*P)

Transfer function:
       4000 s^2 + 60000 s + 200000
-----------------------------------------
s^4 + 30 s^3 + 4325 s^2 + 61500 s + 202500

step(100*H)
diary off
```

5c

Heater temperature $z(t)$ in response to $x(t) = 100u(t)$.

$K = 61.25$, $b = 1$



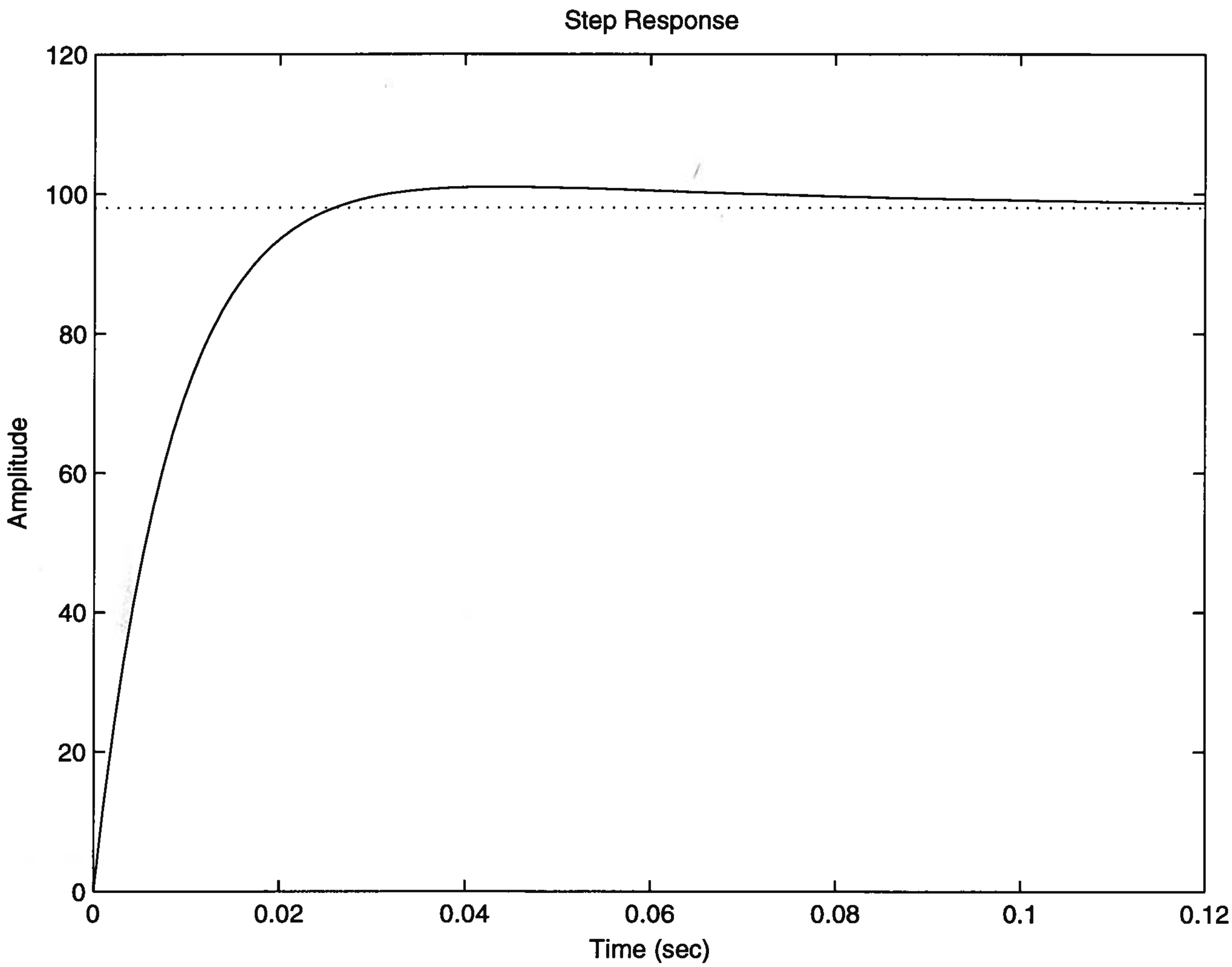
```
diary on
s=tf('s')

Transfer function:
s

P=40/(s^2+15*s+50)

Transfer function:
      40
---------------
s^2 + 15 s + 50

H=(K)/(1+K*P)

Transfer function:
100 s^2 + 1500 s + 5000
-----------------------
   s^2 + 15 s + 4050

step(100*H)
diary off
s=tf('s')

Transfer function:
s

P=40/(s^2+15*s+50)

Transfer function:
      40
---------------
s^2 + 15 s + 50

K=61.25

K =

   61.2500

H=(K)/(1+K*P)

Transfer function:
61.25 s^2 + 918.8 s + 3062
--------------------------
    s^2 + 15 s + 2500

step(100*H)
diary off
```

5d

Response of $w(t)$ to

$X(t) = 100\,u(-1)$,

$k = 61.25$, $G(s) = 1 + 0.05\,s$

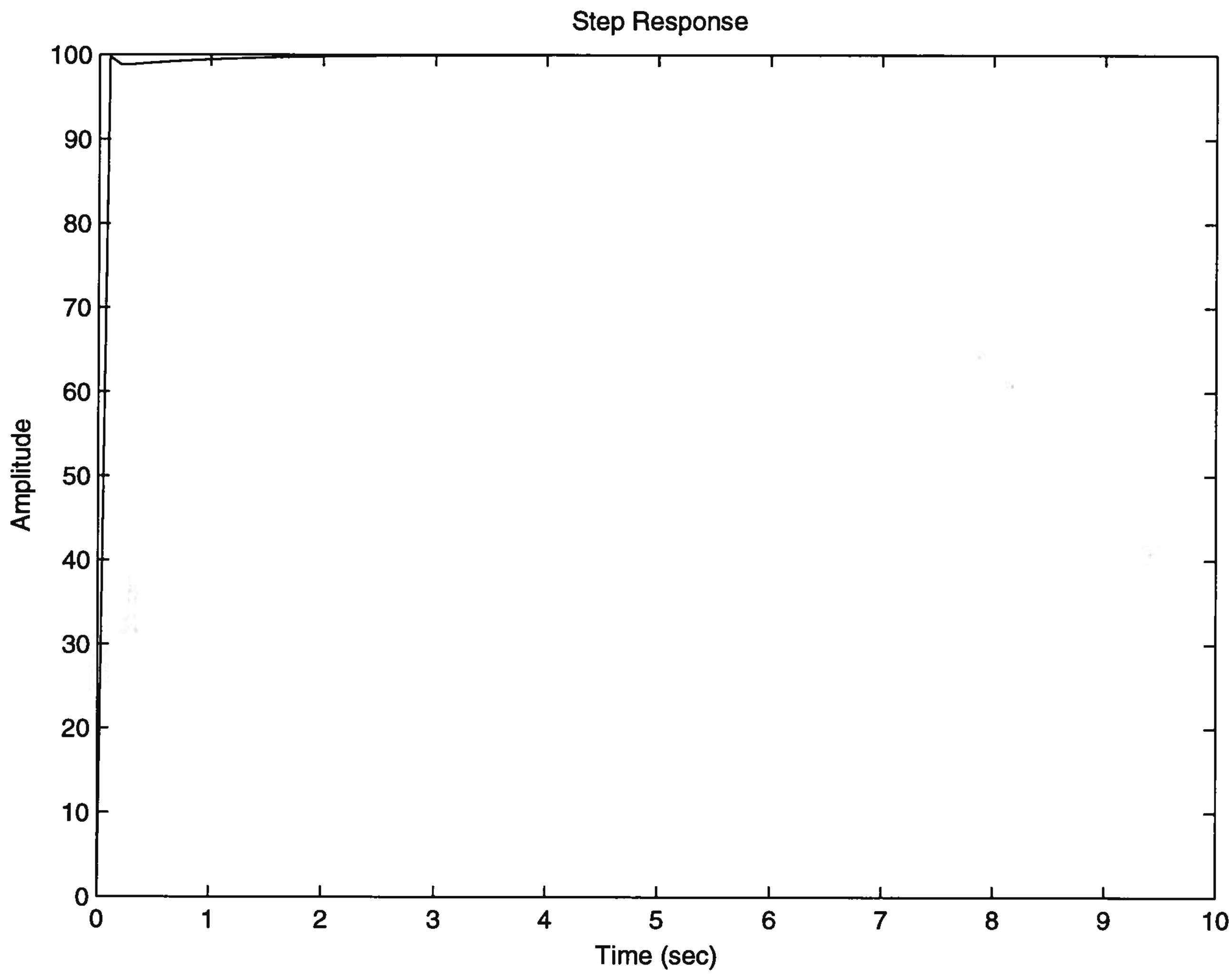


```
diary on
s=tf('s')

Transfer function:
s

P=40/(s^2+15*s+50)

Transfer function:
      40
---------------
s^2 + 15 s + 50

G=(1+0.05*s)

Transfer function:
0.05 s + 1

K=61.25

K =

   61.2500

H=K*G*P/(1+K*G*P)

Transfer function:
   122.5 s^3 + 4288 s^2 + 42875 s + 122500
---------------------------------------------
s^4 + 152.5 s^3 + 4612 s^2 + 44375 s + 125000

step(100*H)
diary off
```

5e

Response of $w(t)$ to $x(t) = 100u(t)$,

$k = 61.25$, $G(s) = 1 + 0.05s + 1/s$



```
diary on
s=tf('s')

Transfer function:
s

P=40/(s^2+15*s+50)

Transfer function:
      40
---------------
s^2 + 15 s + 50

G=(1+0.05*s+1/s)

Transfer function:
0.05 s^2 + s + 1
----------------
       s

K=61.25

K =

   61.2500

H=K*G*P/(1+K*G*P)

Transfer function:
   122.5 s^5 + 4288 s^4 + 45325 s^3 + 159250 s^2 + 122500 s
--------------------------------------------------------------
s^6 + 152.5 s^5 + 4612 s^4 + 46825 s^3 + 161750 s^2 + 122500 s

step(100*H)
diary off
```