3

Dan Ports
6.003 PS3

3.24. a. $a_0 = \frac{1}{2}\left[\int_0^1 t\,dt + \int_1^2 2-t\,dt\right] = \frac{1}{2}\left[\frac{1}{2}+\frac{1}{2}\right]$
$= \boxed{\frac{1}{2}}$

b. $\frac{dx}{dt} = \begin{cases} 1, & 0 < t < 1 \\ -1, & 1 < t < 2 \end{cases}$

We note that the square wave that has value 1/2 from $t=0$ to $t=2$ and $-1/2$ from $t=2$ to $t=4$ has Fourier series coefficients

$$d_k = \begin{cases} \frac{\sin k\pi/2}{k\pi} e^{-jk\pi/2}, & k \neq 0 \\ 0, & k=0 \end{cases}$$

(Oppenheim & Willsky, p. 207).

The function dx/dt is simply a time-scaled and scaled version of that square wave,

so $\frac{dx}{dt} = \sum_{k=-\infty}^{\infty} b_k e^{jk\pi t}$,

with $b_k = 2d_k = \begin{cases} \frac{2}{k\pi}\sin(k\pi/2)\, e^{-jk\pi/2}, & k \neq 0 \\ 0, & k=0 \end{cases}$

$$\boxed{= \begin{cases} \frac{-2}{k\pi} j, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}}$$

c. $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\pi t}$

$a_0 = 1/2$, $a_k = \frac{1}{jk2\pi/2} b_k = \begin{cases} \frac{-2j}{jk^2\pi^2}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}$

$$\boxed{a_k = \begin{cases} \frac{-2}{k^2\pi^2}, & k \text{ odd} \\ 0, & k \text{ even},\ k \neq 0 \\ 1/2, & k=0 \end{cases}}$$

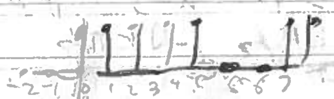
3.26 a. a_0 is real, but $a_{-k} = a_k$, not a_k^*. This means the Fourier coefficients aren't conjugate-symmetric, so $x(t)$ is not real.

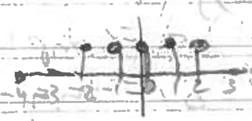
b. The coefficients are even, so $x(t)$ is even from the time-shift property.

c. $\frac{dx}{dt} \Rightarrow \begin{cases} 0, & k=0 \\ jk\omega_0 (1/2)^{|k|} = -k\omega_0 (1/2)^{|k|}, & k \neq 0 \end{cases}$

$x(t)$ is not even because the coefficients aren't even.

3.28. Tables of numerical coefficients and plots of magnitude / phase are attached on a separate page

a. 

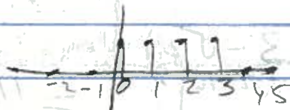
This is a time-shift of the signal , which from O&W p. 218 has Fourier coefficients

$$b_k = \begin{cases} 5/7, & k=0 \\ \frac{1}{7}\frac{\sin[2\pi k(2+1/2)/7]}{\sin(\pi k/7)}, & k \neq 0. \end{cases}$$

We time-shift this signal to the right by 2, so $a_k = b_k e^{-jk2\pi/7 \cdot 2}$

$$\boxed{a_k = \begin{cases} 5/7, & k=0 \\ \frac{1}{7}\frac{\sin(5\pi k/7)}{\sin(\pi k/7)} e^{-jk4\pi/7}, & k \neq 0 \end{cases}}$$

3.28 a. ii



$a_0 = 4/6 = \boxed{2/3}$

$a_1 = 1/6\,(1 + e^{j2\pi/6} + e^{j4\pi/6} + e^{j6\pi/6}) = 1/6\,(1 + \frac{1}{2} + \frac{\sqrt{3}}{2}j - \frac{1}{2} + \frac{\sqrt{3}}{2}j - 1) = \boxed{\frac{\sqrt{3}j}{6}}$

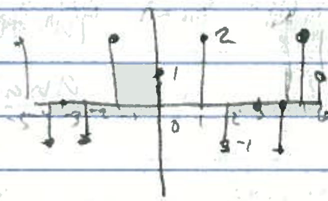
$a_2 = 1/6\,(1 + e^{j4\pi/6} + e^{j8\pi/6} + e^{j10\pi/6}) = \boxed{1/6}$

$a_3 = 1/6\,(1 + e^{j6\pi/6} + e^{j12\pi/6} + e^{j18\pi/6}) = (1 + -1 + 1 + -1) = \boxed{0}$

$a_4 = 1/6\,(1 + e^{j8\pi/6} + e^{j16\pi/6} + e^{j24\pi/6}) = (1 + -1/2 - \frac{\sqrt{3}}{2}j + -\frac{1}{2} - \frac{\sqrt{3}}{2}j + 1) = \boxed{1/6}$

$a_5 = 1/6\,(1 + e^{j10\pi/6} + e^{j20\pi/6} + e^{j30\pi/6}) = (1 + 1/2 - \frac{\sqrt{3}}{2}j - \frac{1}{2} - \frac{\sqrt{3}}{2}j + 1) = \boxed{-\frac{\sqrt{3}}{6}j}$

iii.



$\boxed{a_0 = 1/2}$

for $k \neq 0$, $a_k = 1/6\,(1 + 2e^{j2\pi/6 k} - e^{j4\pi/6 k} - e^{j8\pi/6 k} + 2e^{j10\pi/6 k})$

$= 1/6\,(1 + 2(e^{j\pi/3 k} + e^{-j\pi/3 k}) - (e^{j2\pi/3 k} + e^{-j2\pi/3 k})$

$\boxed{= 1/6\,(1 + 4\cos(\frac{\pi}{3}k) - 2\cos(\frac{2\pi}{3}k))}$

b. $x[n] = \sin(2\pi n/3)\cos(\pi n/2)$

$= \frac{1}{2j}(e^{2\pi n/3} - e^{-2\pi n/3}) \cdot 1/2\,(e^{\pi n/2} + e^{-\pi n/2})$

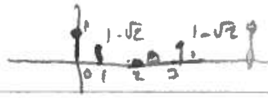
$= 1/4j\,(e^{jn7\pi/6} + e^{jn\pi/6} - e^{-\pi n/6} - e^{-7\pi/6})$

$= 1/4j\,(e^{jw_0 n7} + e^{jw_0 n} - e^{-jw_0 n} - e^{-jw_0 n7})$

This is the expression for a Fourier series whose coefficients from -0 to 11 are

$$\boxed{\begin{aligned} a_k &= 1/4j \text{ for } k=1,\ k=7 \\ &= -1/4j \text{ for } k=5,\ k=11 \end{aligned}}$$

3.28 c.



$a_0 = 1/4(1 + 2(1-\sqrt{2})) = \boxed{1/4\,(3-\sqrt{2})}$

$a_1 = 1/4[1 + (1-\sqrt{2})(e^{-j2\pi/4} + e^{-3j2\pi/4})] = \boxed{1/4}$

$a_2 = 1/4[1 + (1-\sqrt{2})(e^{-j4\pi/4} + e^{-6j2\pi/4})] =$

$= 1/4[1 + (1-\sqrt{2})\cdot -2) = \boxed{1/4(2\sqrt{2} - 1)}$

$a_3 = 1/4(1 + (1-\sqrt{2})\cdot(e^{-j\,\pi/4} + e^{-j8\pi/4}) = \boxed{1/4}$

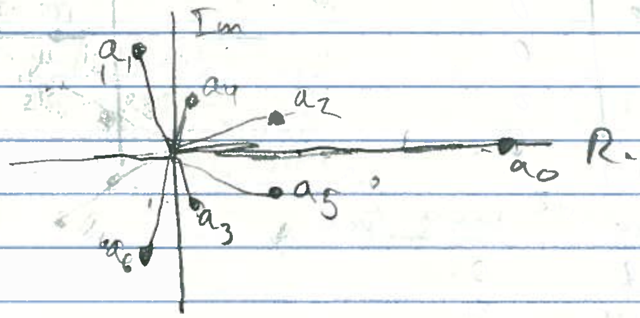
d. $a_0 = 11 - \sqrt{2}$
$a_1 = -\sqrt{6} - 2$
$a_2 = \sqrt{2} + 2$
$a_3 = 1$
$a_4 = 2 - \sqrt{2}$
$a_5 = \sqrt{6} - 2$
$a_6 = \sqrt{2} - 1$
$a_7 = \sqrt{6} - 2$
$a_8 = 2 - \sqrt{2}$
$a_9 = 1$
$a_{10} = \sqrt{2} + 2$
$a_{11} = -\sqrt{6} - \sqrt{2}$

(coefficients obtained by evaluating the summation in the discrete Fourier analysis eqn)

Graphs of magnitude & phase

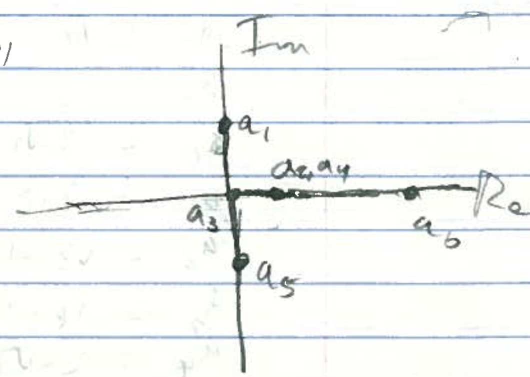
a.i.

index	value	magnitude	phase (rad)
a_0	$= 5/7$	.714	0
a_1	$= -.057 + .251j$	.257	1.80
a_2	$= .160 - .077j$	.178	0.449
a_3	$= .040 + .050j$	.064	-0.898
a_4	$= .040 - .050j$	.064	0.898
a_5	$= .160 + 0.077j$	.178	-0.449
a_6	$= -0.57 + .251j$	.257	-1.80



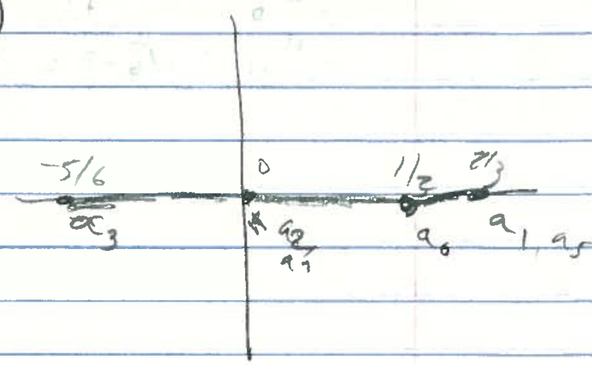
ii.

index	value	magnitude	phase (rad)
a_0	$= 2/3$	$2/3$	0
a_1	$= \sqrt{3}j/6$	$\sqrt{3}/6$	$\pi/2$
a_2	$= 1/6$	$1/6$	0
a_3	$= 0$	0	0
a_4	$= 1/6$	0	0
a_5	$= -\sqrt{3}/6j$	$\sqrt{3}/6$	$-\pi/2$



iii.

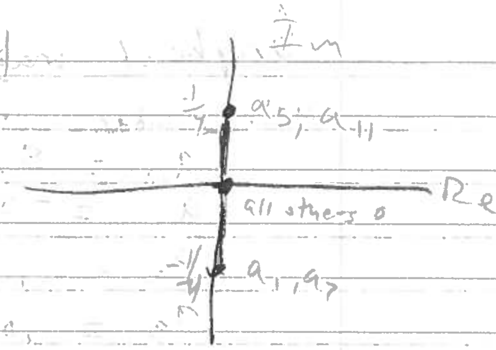
index	value	mag	phase (rad)
a_0	$= 1/2$	$1/2$	0
a_1	$= 2/3$	$2/3$	0
a_2	$= 0$	0	0
a_3	$= -5/6$	$5/6$	π
a_4	$= 0$	0	0
a_5	$= 2/3$	$2/3$	0



b.

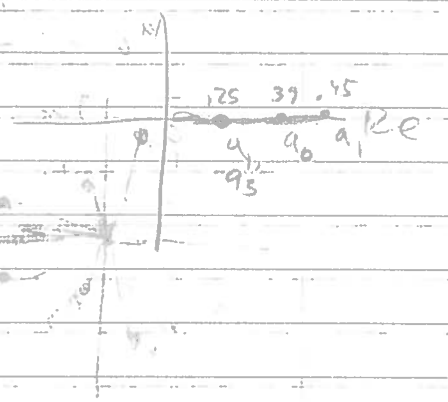
val	mag	phase (rad)
$a_0 = 0$	0	0
$a_1 = 1/4j$	1/4	$-\pi$
$a_5 = -1/4j$	1/4	π
$a_7 = 1/4j$	1/4	$-\pi$
$a_{11} = -1/4j$	1/4	π

all other coefficients zero



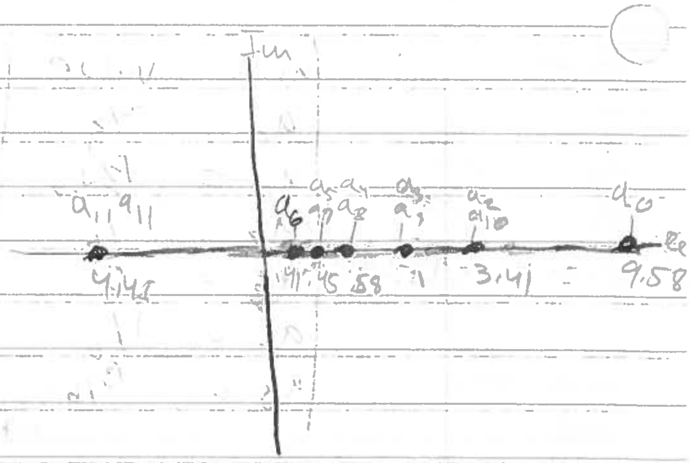
c.

val	mag	Phase (rad)
$a_0 = .396$	.396	0
$a_1 = 1/4$	.25	0
$a_2 = .457$	.457	0
$a_3 = 1/4$	.25	0



d.

val	mag	Phase (rad)
$a_0 = 11 - \sqrt{2}$	9.58	0
$a_1 = -\sqrt{6} - 2$	4.415	π
$a_2 = \sqrt{2} + 2$	3.41	0
$a_3 = 1$	1	0
$a_4 = 2 - \sqrt{2}$	.586	0
$a_5 = \sqrt{6} - 2$	.449	0
$a_6 = \sqrt{2} - 1$	.414	0
$a_7 = \sqrt{6} - 2$	.449	0
$a_8 = 2 - \sqrt{2}$	.586	0
$a_9 = 1$	1	0
$a_{10} = \sqrt{2} + 2$	3.41	0
$a_{11} = -\sqrt{6} - 2$	4.45	π



3.36

a. $y[n] - 1/4\, y[n-1] = x[n]$

Suppose $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega}) e^{j\omega n}$:

$H(e^{j\omega}) e^{j\omega n} - 1/4\, H(e^{j\omega}) e^{j\omega(n-1)} = e^{j\omega n}$

$H(e^{j\omega}) (1 - 1/4\, e^{-j\omega}) = 1$

$\Rightarrow H(e^{j\omega}) = \dfrac{1}{1 - 1/4\, e^{-j\omega}}$

a. Write $\sin(3\pi/4\, n)$ as a set of eigenfcns:

$x[n] = \sin(3\pi/4\, n) = \frac{1}{2j}\left(e^{j3\pi/4 n} - e^{-j3\pi/4 n}\right)$

So $y[n] = \frac{1}{2j}\left[H(e^{j3\pi/4}) e^{j3\pi/4 n} - H(e^{-j3\pi/4}) e^{-j3\pi/4 n}\right]$

$$\boxed{y[n] = \frac{1}{2j}\left[\frac{1}{1 - 1/4\, e^{-j3\pi/4}} e^{j3\pi/4 n} + \frac{j\,1}{1 - 1/4\, e^{j3\pi/4}} e^{-j3\pi/4 n}\right]}$$

b. Write $\cos \pi/4\, n + 2\cos \pi/2\, n$ as a sum of eigenfcns:

$x[n] = \cos(\pi/4\, n) + 2\cos(\pi/2\, n)$

$= \frac{1}{2}\left(e^{j\pi/4 n} + e^{-j\pi/4 n} + 2e^{j\pi/2 n} + 2e^{-j\pi/2 n}\right)$

So $y[n]$ is a sum of eigenvals times eigenfunctions:

$y[n] = \frac{1}{2}\left[H(j\pi/4) e^{j\pi/4 n} + H(-j\pi/4) e^{-j\pi/4 n} + 2H(j\pi/2) e^{j\pi/2} + 2H(-j\pi/2) e^{-j\pi/2}\right]$

$$\boxed{y[n] = \frac{1}{2}\left[\frac{1}{1 - 1/4\, e^{-j\pi/4}} e^{j\pi/4 n} + \frac{1}{1 - 1/4\, e^{j\pi/4}} e^{-j\pi/4 n} + \frac{2}{1 - 1/4\, e^{-j\pi/2}} e^{j\pi/2 n} + \frac{2}{1 - 1/4\, e^{j\pi/2}} e^{-j\pi/2 n}\right]}$$