Dan Porzio 3/3
6.003 PS5

1. f. $\frac{\sin \pi t}{\pi t}$ has Fourier transform $\begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases}$

$$\frac{\sin 2\pi t}{\pi t} \xrightarrow{\mathcal{F}} \begin{cases} 1, & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases}$$

$$\Rightarrow \frac{\sin 2\pi(t-1)}{\pi(t-1)} \xrightarrow{\mathcal{F}} \begin{cases} e^{-j\omega} & |\omega| < 2\pi \\ 0 & |\omega| > 2\pi \end{cases}$$

Multiplying these leads to convolution in the Fourier domain (and a $1/2\pi$ factor). Note that $\frac{\sin \pi t}{\pi t}$'s Fourier transform is $u(t+\pi) - u(t-\pi)$. Differentiating it gives $\delta(t+\pi) - \delta(t-\pi)$. We can convolve this with $\frac{\sin 2\pi(t-1)}{\pi(t-1)}$'s Fourier transform, which is $e^{-j\omega}\left[u(t+2\pi) - u(t-2\pi)\right]$. Convolving with deltas gives us:

$$e^{-j(\omega+\pi)}\left[u(t+2\pi+\pi) - u(t-2\pi+\pi)\right]$$
$$- e^{-j(\omega-\pi)}\left[u(t+2\pi-\pi) - u(t-\pi-2\pi)\right]$$
$$= -e^{-j\omega}\left[u(t+3\pi) - u(t-\pi)\right]$$
$$+ e^{-j\omega}\left[u(t+\pi) - u(t-3\pi)\right]$$
$$= e^{-j\omega}\left[-u(t+3\pi) + u(t+\pi) + u(t-\pi) - u(t-3\pi)\right]$$

Integrating and dividing gives:

$$X(j\omega) = \begin{cases} -1/2\pi\left(\frac{e^{-j\omega}}{-j} + 1\right) & -3\pi < \omega < -\pi \end{cases}$$

h. Note that $x(t)$ has derivative

$$-\delta(t+2) + u(t+1) - u(t-1) + \delta(t-2).$$

This is a pair of deltas and a square wave. It has Fourier transform

$$e^{jw2} + \frac{2\sin w}{w} + e^{-jw2}$$

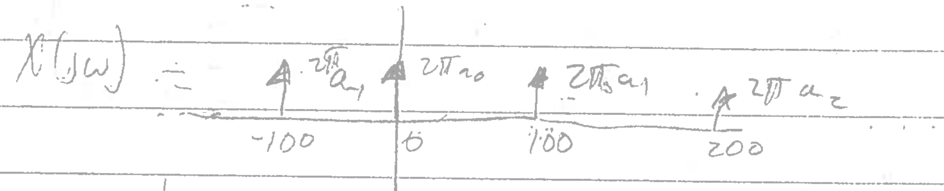
We use the integration property to find $x(t)$'s Fourier transform

$$\boxed{X(jw) = \frac{1}{jw}\left[-e^{2jw} + \frac{2\sin w}{w} + e^{-2jw}\right]}$$

2a: $g_1 = \{[\cos(\omega_0 t)\, x(t)]\} * h(t)$.

$\cos(\omega_0 t)$ has Fourier transform $\pi[\delta(t+\omega_0) + \delta(t-\omega_0)]$, multiplying by $x(t)$ corresponds to convolution, $x(t)$ is periodic w/ Fourier coefficients a_k,

so $X(j\omega) =$



Our desired output is

$G_1(j\omega) = 2\pi\, Re[a_5]\,\delta(\omega)$. To achieve this, note that $Re[a_5] = [a_5 + a_5^*]/2 = \frac{1}{2}(a_5 + a_{-5})$ since $x(t)$ is even. The convolution by the two deltas gives us shifted copies of $X(j\omega)$. To get an impulse of the appropriate size at $t=0$, use $\omega_0 = \pm 500$. Then the convolution gives us $\frac{\pi}{2\pi}[2\pi a_{-5} + 2\pi a_5]\delta(\omega)$

$= 2\pi\, Re(a_5)\,\delta(\omega)$, plus various other impulses at intervals of 100. So we require $H(j\omega)$ to be 1 at $\omega = 0$ and 0 at $\omega = 100k$, for integer $k \neq 0$.

Next consider the g_2 constraint.

$\sin(\omega_0 t) \xrightarrow{\mathcal{F}} j\pi[\delta(t-\omega_0) - \delta(t+\omega_0)]$.

$x(t)$ and $X(j\omega)$ are as before. Our desired output is $g_2(t) = Im(a_5) \Rightarrow G_2(j\omega) = 2\pi a_5 \delta(\omega)$.

From before we know $\omega_0 = +500$ or $\omega_0 = -500$.
Try each of these. If $\omega_0 = +500$ then (ignoring impulses at non-zero frequency), the convolution gives $\frac{\pi}{2\pi}(2\pi a_{-5} - 2\pi a_{5})\delta(\omega)$ which is incorrect. For $\omega_0 = -500$, it gives $\pi/2\pi\,(2\pi a_5 - 2\pi a_{-5})\,\delta(\omega)$ $= 2\pi(a_5 - a_{-5})/2 = 2\pi\,\mathrm{Im}\{a_5\}\,\delta(\omega)$, plus other impulses.

So we need $\omega_0 = -500$

and $H(j\omega) = \begin{cases} 1 & \text{at } \omega = 0 \\ 0 & \text{at } \omega = 100k \quad \text{for nonzero integer } k \end{cases}$

b) One such $H(j\omega)$ is

$$H(j\omega) = \begin{cases} 1, & |\omega| < 50 \\ 0, & |\omega| > 50 \end{cases}$$

It has inverse Fourier transform

$$h(t) = \frac{\sin 50t}{\pi t}$$

3. The sinusoid given can be expressed as $\cos\left(\frac{2\pi}{T}t\right) = \left(e^{2\pi/T t} + e^{-2\pi/T t}\right)/2$.

It has Fourier series coefficients

$$a_k = \begin{cases} 1/2, & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

w/ $\omega_0 = 2\pi/T$.

The square wave has Fourier series coefficients

$$b_k = \frac{\sin k\omega_0 T/2}{k\pi} = \frac{\sin k\pi}{k\pi}$$

So the rectified square wave has Fourier coefficients given by the convolution of a_k and b_k.

$$= \frac{\cos\frac{k\pi}{2}}{(1-k^2)\pi} ?$$

$$c_k = \frac{1}{2}\left[\frac{\sin[(k-1)\pi]}{(k-1)\pi} + \frac{\sin[(k+1)\pi]}{(k+1)\pi}\right]$$

~~since multiplication corresponds to convolution,~~

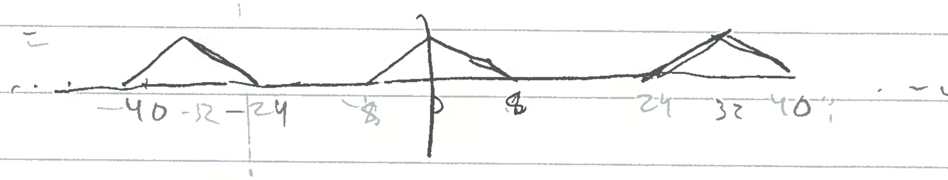
We use this to construct the Fourier transform of the rectified sin wave:

$$C(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} c_k\, \delta(\omega - k\,2\pi/T)$$

w/ c_k as above.

4. The diode circuit rectifies the input signal, since no current is drawn by the bandpass filter. Thus,

$z(t) =$



Note that $z(t)$ is periodic w/ period $\pi/16$, so the Fourier transform consists of impulses at multiples of $\pi/16$. The bandpass filter will cut out all of these except the ones at $\omega = \pm \pi/8$. Further note that $z(t)$ is real and even, so $a_2 = a_{-2}$. We need only find a_2, therefore

$$a_{-2} = a_2 = \frac{1}{32}\int_{-8}^{8} (1-|t|)\, e^{-j \cdot 2 \cdot \frac{2\pi}{32} t}\, dt$$

$$= 8/\pi^2$$

So
$$y(t) = 8/\pi^2 e^{j2\omega_0 t} + 8/\pi^2 e^{-j2\omega_0 t}$$
$$= 16/\pi^2 \cos(\pi/8\, t),$$

$y(t) = \frac{2}{\pi^2} \cos \frac{\pi}{8} t$

5. a. $\boxed{X(e^{j\omega}) = e^{j\omega} + 1 + e^{-j\omega}}$

b. $x[n] = 3^n u[-n-1] = 3^n u[-n] - \delta[n]$

$= \underbrace{(1/3)^{-n} u[-n]}\ - \delta[n]$

Note that this term is the time-reversal of $(1/3)^n u[n]$, which has Fourier transform $\frac{1}{1-1/3 e^{-j\omega}}$. The time-reversal gives $\frac{1}{1-1/3 e^{j\omega}}$, and $\delta[n] \Rightarrow 1$, so by linearity,

$\boxed{X(e^{j\omega}) = \frac{1}{1-1/3 e^{j\omega}} - 1}$

c. $x[n] = (1/3)^{|n|} = (1/3)^n u[n] + (1/3)^{-n} u[-n] - \delta[n]$

We have above the Fourier transforms of each term, so

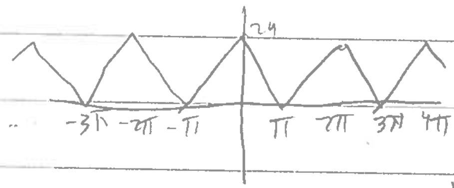
$\boxed{X(e^{j\omega}) = \frac{1}{1-1/3 e^{-j\omega}} + \frac{1}{1-1/3 e^{j\omega}} - 1}$

d. $x[n] = \cos(7\pi n/3)$

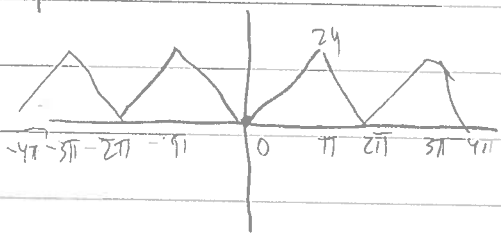
$\boxed{\Rightarrow X(e^{j\omega}) = \pi \sum_{l=-\infty}^{\infty} \left[\delta(\omega - 7\pi/3 - 2\pi l) + \delta(\omega + 7\pi/3 - 2\pi l)\right]}$

6. First note that if $f[n] \overset{\mathcal{F}}{\to} F(e^{j\omega})$, $f[n-n_0] \overset{\mathcal{F}}{\to} e^{-j\omega n_0} X(e^{j\omega})$, so by duality of the Transform, $e^{j\omega_0 n} f[n] \overset{\mathcal{F}}{\to} X(e^{j(\omega-\omega_0)})$. This can also be seen using the DTFT of $e^{j\omega_0 n}$ and applying the multiplication property. Since $-1 = e^{j\pi}$, $(-1)^n = e^{j\pi n}$, and so multiplying by $(-1)^n$ shifts the frequency response to the right by π. Thus:

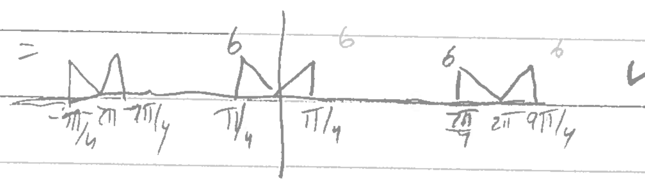
$X(e^{j\omega})$



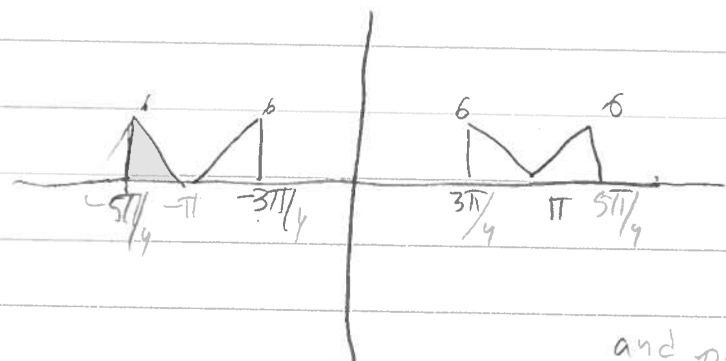
$W(e^{j\omega}) = X(e^{j(\omega-\pi)}) =$



$V(e^{j\omega}) = W(e^{j\omega}) H(e^{j\omega}) =$



$Y(e^{j\omega}) = V(e^{j(\omega-\pi)})$



and periodic w/