Dan Ports

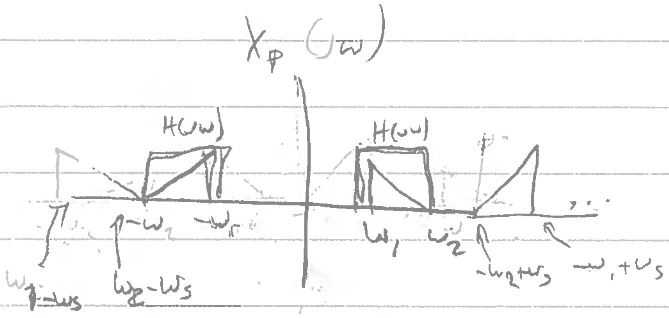
6.003 PS6

1\. 7.26.

$$p(t) = \sum_n \delta(t - nT)$$

$$\Rightarrow P(j\omega) = \frac{2\pi}{T} \sum_k \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$\cancel{X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)}$$



We must ensure that these do not overlap:

$$\omega_2 < -\omega_2 + \omega_s$$

$$\Rightarrow \Rightarrow \omega_s > 2\omega_2$$

So the minimum value of $\omega_s = 2\omega_2$

$$\Rightarrow \boxed{\text{max value of } T = \frac{2\pi}{2\omega_2} = \pi/\omega_2}$$

We apply a filter that keeps only the original signal: $\boxed{\omega_a = \omega_1, \quad \omega_b = \omega_2}$ ✓

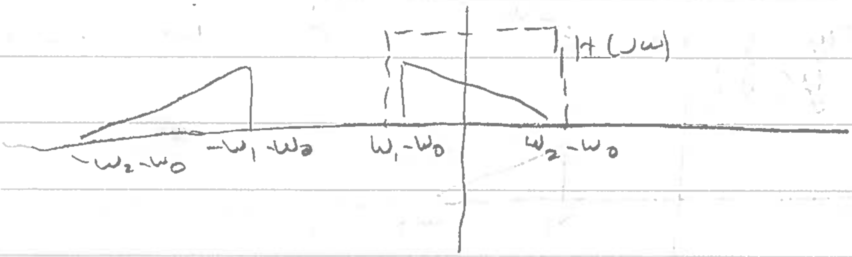
The amplitude is $1 \cdot \frac{2\pi}{T} \cdot \frac{1}{2\pi} \cdot A$; we want it to be 1

$$\Rightarrow \boxed{A = T}$$ ✓

2. 7.27

$$x(t) \longrightarrow \otimes_{\,e^{-j\omega_0 t}} \xrightarrow{x_a} \boxed{H(j\omega)} \longrightarrow \otimes_{\,p(t)} \longrightarrow x_p(t)$$

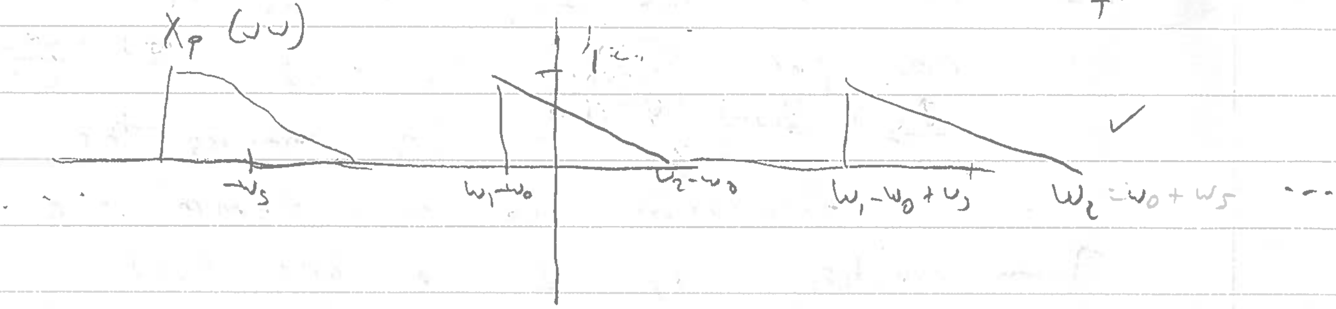
a. $X_a(j\omega) = \frac{1}{2\pi} X(j\omega) * \mathcal{F}(e^{-j\omega_0 t}) = X(j\omega + \omega_0)$



So applying $H(j\omega)$ preserves only the △ shape;

Multiplying by $p(t)$ is a convolution by an impulse train!

$\omega_s = \frac{2\pi}{T}$



b. We must ensure the triangles do not overlap

$\omega_2 - \cancel{\omega_0} < \omega_1 - \cancel{\omega_0} + \omega_s \Rightarrow \omega_s > \omega_2 - \omega_1$

$\omega_2 - \omega_0 - \omega_s < \omega_1 - \omega_0$

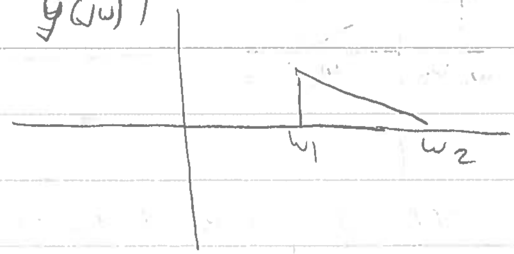
$\Rightarrow \frac{2\pi}{T} > \omega_2 - \omega_1 \Rightarrow T < \frac{2\pi}{\omega_2 - \omega_1}$

So the max val of T is $\boxed{\frac{2\pi}{\omega_2 - \omega_1}}$

✓

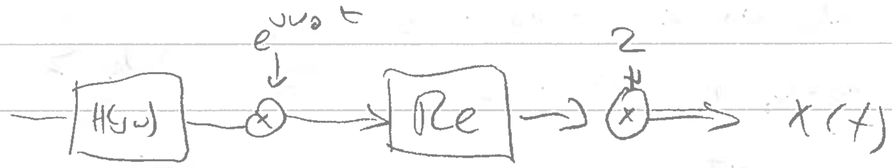
c. To recover $x(t)$, we first discard all but one triangle in the Fourier series by applying $H(\omega)$ again. We then shift it to the right by multiplying by $e^{j\omega_0 t}$. Then we have:

(call this $y(j\omega)$)



We need to add a flipped copy of the triangle between $-\omega_1$ and ω_2. To do this, take the real part since $Re\{y(t)\} = \frac{y(t) + y^*(t)}{2}$

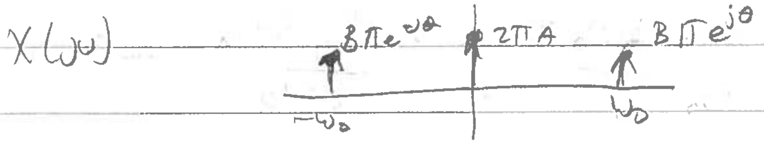
$\xrightarrow{\mathcal{F}} \frac{y(j\omega) + y^*(-j\omega)}{2}$. This makes the Fourier transform conjugate-symmetric. Then multiply by 2 to get $x(t)$.



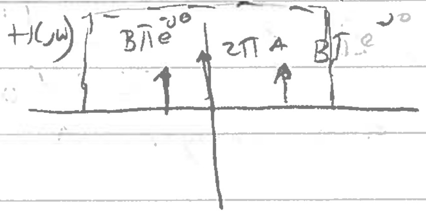
✓

3.7.38. $x(t) = A + B\cos[(2\pi/T)t + \theta]$

Let $\omega_0 = 2\pi/T$ and $\omega_s = 2\pi/(T+\Delta)$

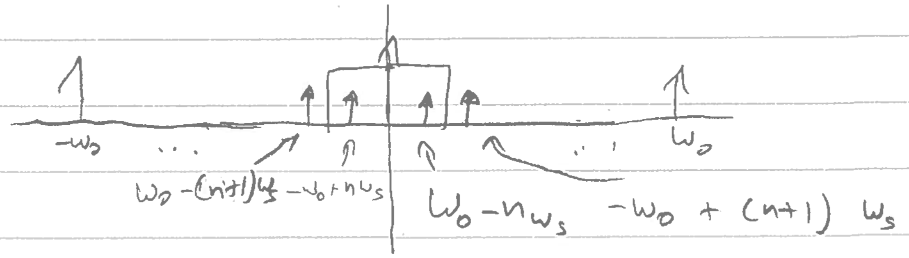


We would like to sample this so we have



within the passing region of $H(\omega)$, but ~~no other impulses~~ passed

For any value of n, we can place restrictions on ω_s such that the nth copy of the impulses lies within the passband of the lowpass and no other does.



That is,

$$\omega_0 - n\omega_s < \frac{1}{2(T+\Delta)} < (n+1)\omega_s - \omega_0$$

$$\omega_0 - (n+1)\omega_s < -\frac{1}{2(T+\Delta)} < -\omega_0 + n\omega_s$$

We also know that, by definition of ω_0 and ω_s and because Δ can only range from 0 to T,

$$\frac{\omega_0}{2} < \omega_s < \omega_0$$

Combining those,

$$\frac{2\pi}{T} - n\frac{2\pi}{T+\Delta} < \frac{1}{2(T+\Delta)} < (n+1)\frac{2\pi}{T+\Delta} - \frac{2\pi}{T}$$

$$\frac{\pi}{T} < \frac{2\pi}{T+\Delta} < \frac{2\pi}{T}$$

$$-\frac{n 2\pi}{T+\Delta} + \omega_0 < \frac{2\pi}{4\pi(T+\Delta)} < (n+1)\frac{2\pi}{T+\Delta} - \omega_0$$

$$-n 2\pi + \omega_0 T + \omega_0 \Delta < 1/2 < (n+1) 2\pi - \omega_0 T - \omega_0 \Delta$$

$$\Delta < \frac{1/2 + 2\pi n - \omega_0 T}{\omega_0} = \frac{1 + 4\pi n}{2\omega_0} - T$$

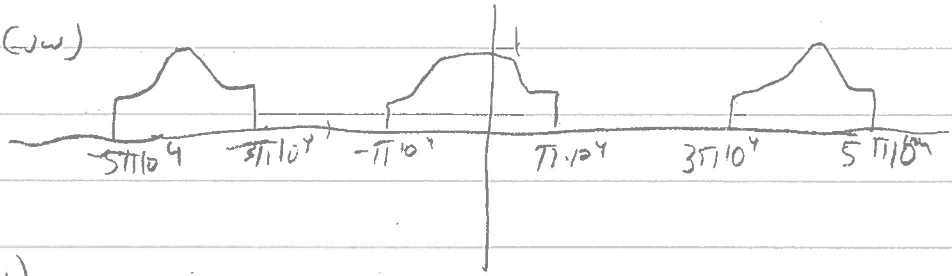
$$\Delta > \frac{2\pi n - \omega_0 T}{\omega_0} = \frac{2\pi n}{\omega_0} - T$$

$$\boxed{\frac{2\pi n}{\omega_0} - T < \Delta < \frac{1 + 4\pi n}{2\omega_0} - T \quad \text{for any integer } n}$$

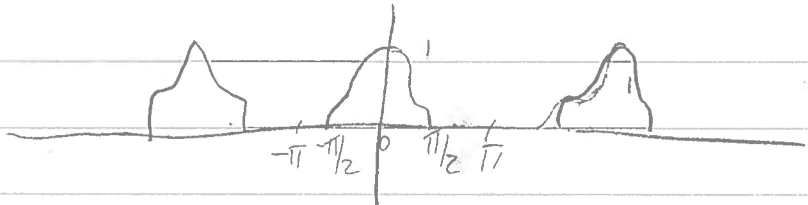
From the graph, $\boxed{a = \frac{\omega_0 - n\omega_s}{\omega_0} = 1 - \frac{nT}{T+\Delta}}$

4. 7.29. $1/T = 20\,kHz \Rightarrow \omega_s = \pi \cdot 4 \times 10^4$

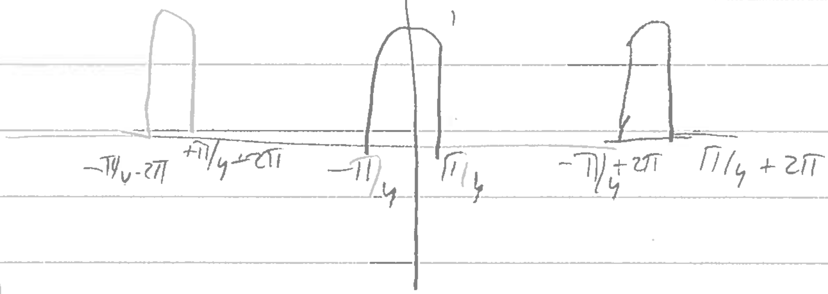
$X_p(j\omega)$



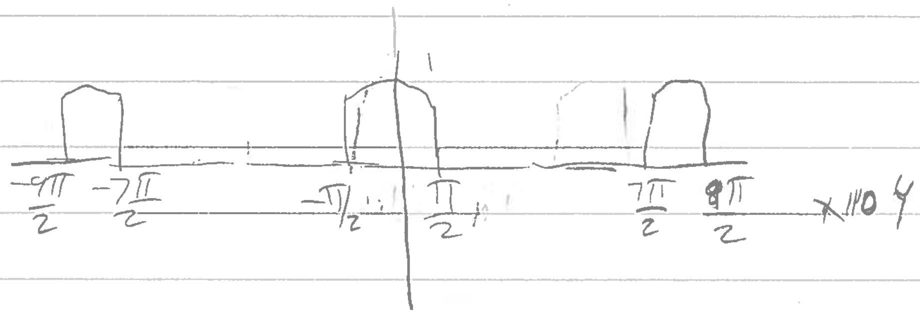
$X(e^{j\omega})$



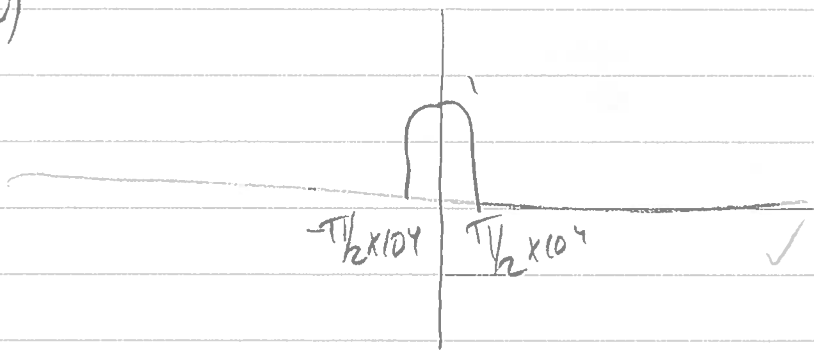
$Y(e^{j\omega})$



$Y_p(e^{j\omega})$



$Y_c(j\omega)$



5. 7.31.

First find the frequency response of the DT filter using the eigenfunction property:

$$H(e^{j\Omega})e^{j\Omega n} = \frac{1}{2}H(e^{j\Omega})e^{j\Omega(n-1)} + e^{j\Omega n}$$

$$H(e^{j\Omega})\left[1 - \tfrac{1}{2}e^{-j\Omega}\right] = 1.$$

$$H(e^{j\Omega}) = \frac{1}{1 - \tfrac{1}{2}e^{-j\Omega}}$$

Then we convert it to a CT filter:

$$\boxed{H(j\omega) = \frac{1}{1 - \tfrac{1}{2}e^{-j\omega T}}}$$

✓

6. 7.41.

$$s(t) = x(t) + \alpha x(t - T_0)$$

$$\Rightarrow S(j\omega) = X(j\omega) + \alpha X(j\omega) e^{-j\omega T_0}$$

a. Let $T = T_0$. We convert the continuous-time sequence to a discrete-time one, apply a DT filter and convert back to CT, so we can treat the system essentially as a CT filter $H(j\omega)$. We want:

$$\left(X(j\omega) + \alpha X(j\omega) e^{-j\omega T_0}\right) H(j\omega) = X(j\omega)$$

$$\Rightarrow H(j\omega) = \frac{1}{1 + \alpha e^{-j\omega T_0}}$$

To convert this to the DT filter, $H(e^{j\Omega})$ we use $\omega = \frac{\Omega}{T}$ $\Rightarrow$

$$H(e^{j\Omega}) = \frac{1}{1 + \alpha e^{-j\Omega T_0 / T}} = \frac{1}{1 + \alpha e^{-j\Omega}}$$

This is produced by the difference eqn $\boxed{y[n] + \alpha y[n-1] = x[n]}$ ✓

b. The multiplication by the impulse train results in an attenuation of $\frac{2\pi}{T} \cdot \frac{1}{2\pi} = \frac{1}{T}$. So we multiply by $\boxed{A = T}$ to give the correct output level. ✓

c. In this case we do not have $T = T_0$. We apply the same strategy as in part a, and get the same result

$$\boxed{H(e^{j\Omega}) = \frac{1}{1+\alpha e^{-j\Omega T_0/T}}}$$

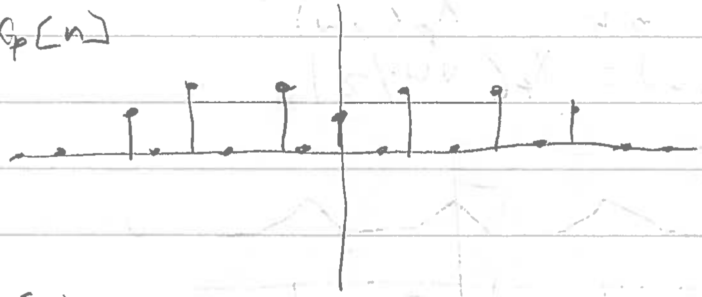
though it cannot be further simplified this time.

Again, $\boxed{A = T}$

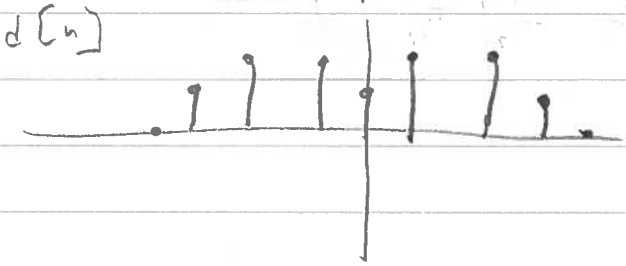
We can choose any period T that satisfies the Nyquist criterion. $\boxed{T < \pi/\omega_m}$

7.7.35

a. $x_p[n]$

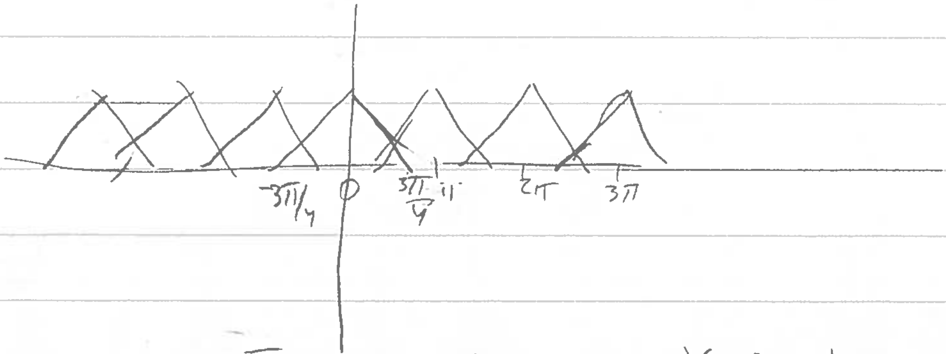


$x_d[n]$



b. $x_p[n] = x[n] \cdot \sum_{k=-\infty} \delta[n-2k]$

$X_p(e^{j\Omega}) = X(e^{j\Omega}) \circledast \frac{2\pi}{2} \sum \delta(\omega - \frac{2\pi k}{2})$



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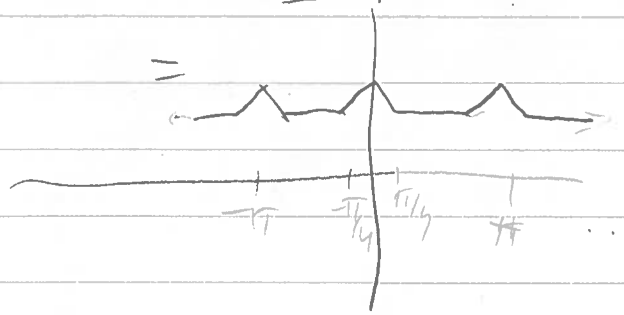
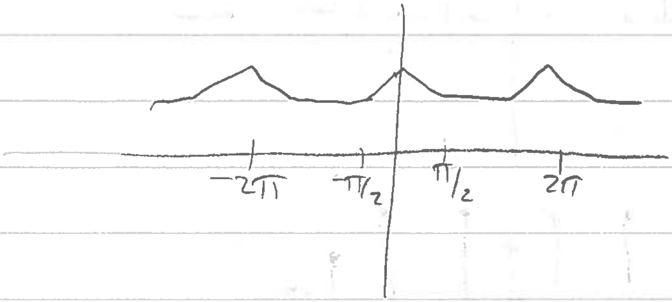
$X_p(j\omega)$



Note that $X_d(w)$ is just a stretched version of $X_p(jw)$:

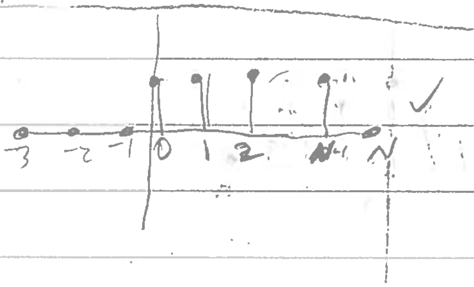
$$X_d(jw) = X_p(jw/2)$$



✓

8. 7.50

a. $\boxed{h_0[n] = \begin{cases} 1, & 0 \le n < N \\ 0 & \text{otherwise} \end{cases}}$



b. The ideal lowpass filter (call this $h_e[n]$) recovers $x[n]$ from $x_p[n]$.

So we want $h_0[n] * h[n] = h_e[n]$

$$\Rightarrow H_0(e^{jw}) \cdot H(e^{jw}) = H_e(e^{jw})$$

$$\Rightarrow H(e^{jw}) = \frac{H_e(e^{jw})}{H_0(e^{jw})}$$

Note that $H_e(e^{jw}) = 1$, $w < w_m$ and 0 otherwise

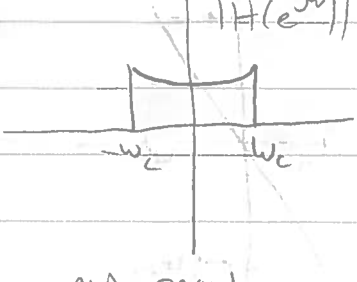
We consider the case $N = 3$. Then

do this for the general (N) case, not for N=

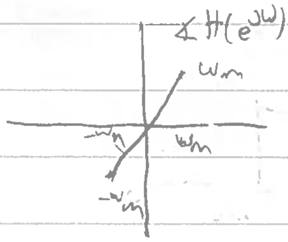
$h_0[n] = \delta[n] + \delta[n-1] + \delta(n-2]$

$$\Rightarrow H_0(e^{jw}) = 1 + e^{-jw} + e^{-2jw}$$

So $\boxed{H(e^{jw}) = \begin{cases} \dfrac{1}{1 + e^{-jw} + e^{-2jw}}, & |w| < w_m \\ 0 & \text{otherwise} \end{cases}}$

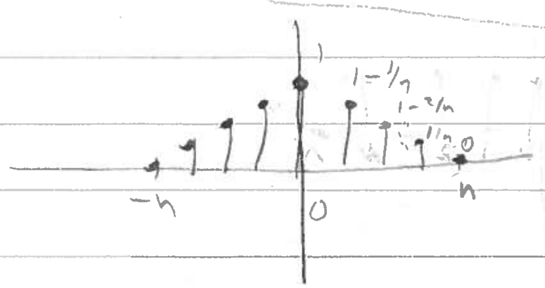


and periodic w/ period 2π



and periodic w/ period 2π

c. $h_1[n] = \begin{cases} 1 - \frac{|n|}{N}, & 0 \le n \le N \\ 0 & \text{otherwise} \end{cases}$



slope is $\pm 1/n$

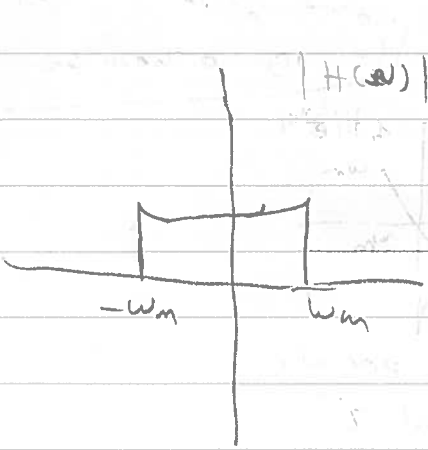
d. Note that $h_1[n] = h_0[n+1] * h_0[n+1]$

so $H_1(j\omega) = [H_0(j\omega) e^{-j\omega}]^2$

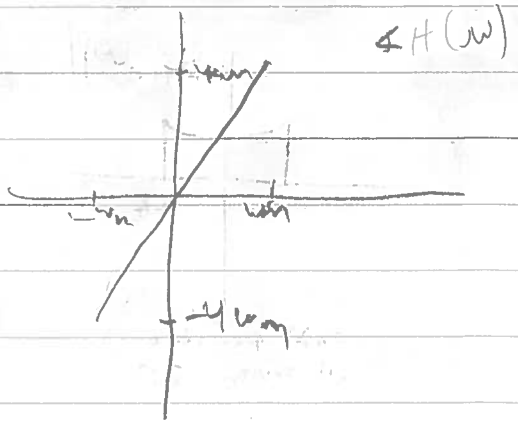
Again we use the relation that

$$H(j\omega) = \frac{H_e(j\omega)}{H_1(j\omega)} = \frac{H_e(j\omega)}{[H_0(j\omega)]^2}$$

$$\boxed{H(j\omega) = \begin{cases} \dfrac{e^{-2j\omega}}{(1 + e^{-j\omega} + e^{-2j\omega})^2}, & |\omega| \le \omega_m \\ 0 & \text{otherwise} \end{cases}}$$



and periodic w/ period 2π



and periodic w/ period 2π