MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Spring 2003

Quiz 1
Thursday, March 13, 2003

Directions: The exam consists of 5 problems on pages 2 to 17. Please make sure you have all the pages. Tables of transforms and properties are supplied to you at the end of this booklet. **Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. DO IT NOW!**. All sketches must be adequately labeled. Unless indicated otherwise, **answers must be derived or explained**, not just simply written down. This examination is closed book, but students may use one 8 1/2 × 11 sheet of paper for reference. Calculators may not be used.

NAME: DAN PORTS

Check your section	Section	Time	Room	Rec. Instr.	TA
☐	1	10-11	36-112	Prof. Daniel	Mario
☐	2	11-12	36-112	Prof. Daniel	Mario
☐	3	12- 1	36-112	Prof. D. Freeman	Salil, Peter
☐	4	1- 2	36-112	Prof. D. Freeman	Salil, Peter
☐	5	10-11	36-114	Prof. W. Freeman	Stephen
☐	6	11-12	34-302	Prof. W. Freeman	Stephen
☒	7	12- 1	34-304	Taka	Siddhartan, Ariel
☐	8	1- 2	34-304	Taka	Siddhartan, Ariel

Please leave the rest of this page blank for use by the graders:

Problem	**No. of points**	**Score**	**Grader**
1	15	14	WMA
2	20	17	WMA
3	20	19	PS
4	25	25	AS
5	20	20	SG
Total	100	95	SMH

14/15

PROBLEM 1 (15%)

Consider the following three systems with a real input signal $x(t)$, and a real output signal $y(t)$.

SYSTEM A:

$$y(t) = \lceil x(t) \rceil$$

where $\lceil \alpha \rceil$ is defined as the smallest integer greater than or equal to α.

SYSTEM B:

$$y(t) = \begin{cases} -1, & \text{if } x(t) < -1 \\ x(t), & \text{if } -1 \leq x(t) \leq 1 \\ 1, & \text{if } x(t) > 1 \end{cases}$$

SYSTEM C:

$$y(t) = \left[\int_0^t x(\tau) d\tau \right] u(t)$$

where $u(t)$ is the step function.

For the following three questions a detailed explanation is not necessary.

+5 **Part a.** Which of the systems above are linear? C only

+4 **Part b.** Which of the systems above are time invariant? A only

+5 **Part c.** Which of the systems above are causal? A, B, C

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Work Page for Problem 1

A. not linear

$x(t) \Rightarrow y_1(t) = \lceil x(t) \rceil$

$x(t-t_0) \Rightarrow y_2(t) = \lceil x(t-t_0) \rceil = y_1(t-t_0)$ TI

causal – doesn't depend on future vals.

B. $x(t) \rightarrow y(t) = 1,\ t>1$

$Ax(t) \rightarrow y(t) = 1,\ t>1$ nonlinear

not TI –

causal

C $x_3(t) = Ax(t) + By(t) \rightarrow \left[\int_0^t [Ax(\tau) + By(\tau)]\, d\tau\right] u(t)$

$= A\left(\int_0^t x(\tau)\right) u(t) + B\left(\int_0^t y(\tau)\, d\tau\right) u(t)$

$= Ay_1(t) + By_2(t)$ Linear

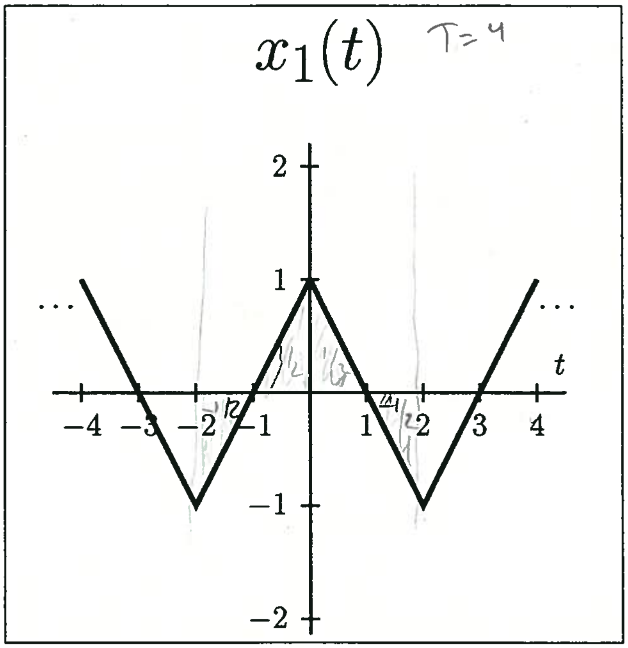
$x_2(t) = x_1(t-t_0) = \left(\int_0^{t} x(\tau - t_0)\, d\tau\right) u(t)$

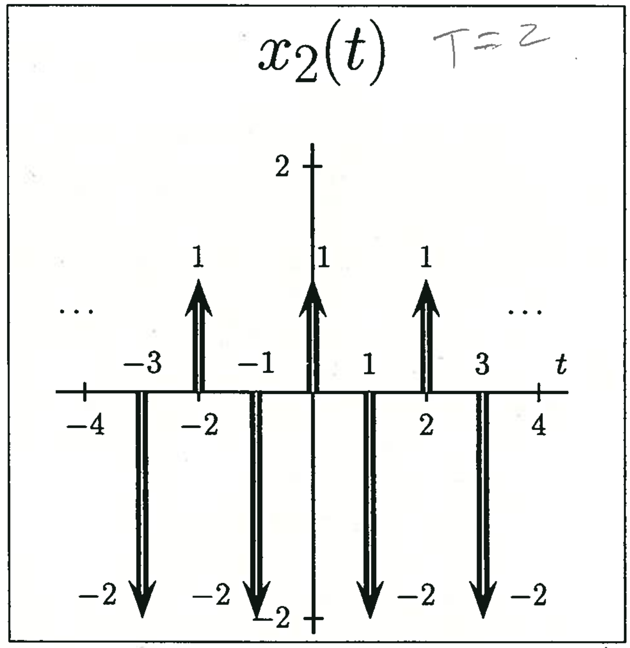
$\neq \left(\int_0^{t-t_0} x(\tau)\, d\tau\right) u(t-t_0)$ not TI

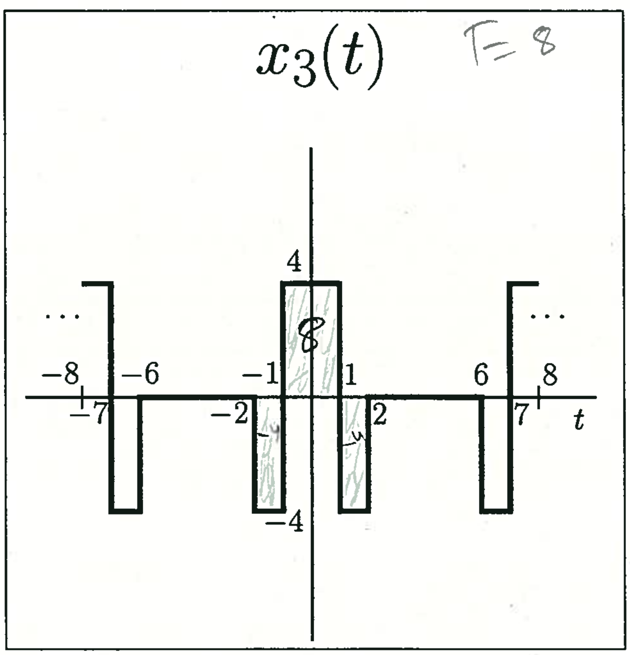
17/20

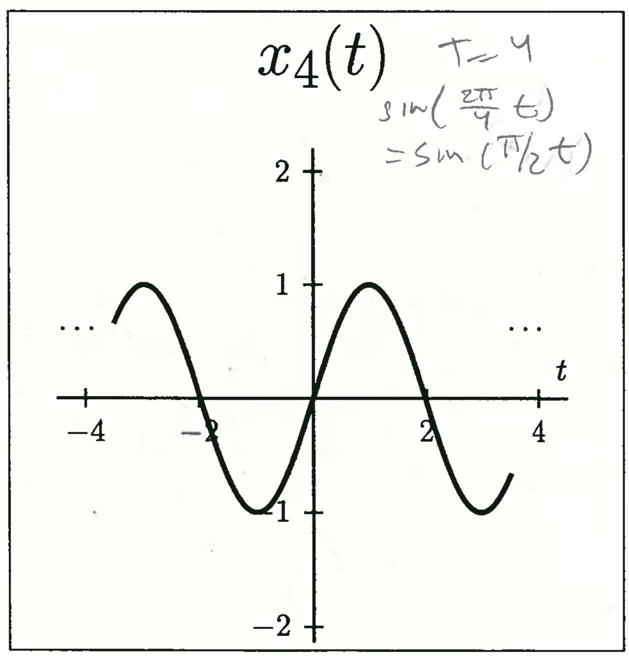
PROBLEM 2 (20%)

Consider the following four *periodic* CT signals.









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The following three questions refer to the Fourier series $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$.

5/5

Part a. What are the values of the Fourier series coefficients a_0 for each of the above four CT signals?

$1/2 + 1/2 - 1/2 - 1/2 = 0$, adding the shaded areas

+1 Signal $x_1(t)$: $a_0 =$ 0

$\frac{1}{T} \int_{-0.5}^{1.5} \delta(t) - 2\delta(t-2) = \frac{1}{2}[1-2] = -1/2$

+2 Signal $x_2(t)$: $a_0 =$ $-1/2$

$8 + -4 + -4 = 0$, adding the shaded areas

+1 Signal $x_3(t)$: $a_0 =$ 0

$\int_0^4 \sin(\pi/2\, t) = 0$

+1 Signal $x_4(t)$: $a_0 =$ 0

7/7

Part b. Which of the four CT signal above has the following Fourier series coefficients a_k?

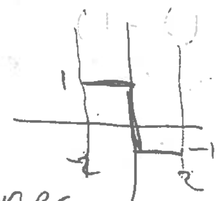
$$a_k = \begin{cases} \frac{4}{\pi^2 k^2} & \text{if } k \text{ is odd} \\ 0 & \text{if } k \text{ is even} \end{cases}$$

real even coefficients $\Rightarrow$ real and even signal

$\Rightarrow$ not x_4

$a_0 = 0 \Rightarrow$ not x_2

Observe that x_1 is the integration of the square wave w/ period 4:



This square wave has Fourier series
$b_0 = 0$, $b_k = \frac{2\sin(\frac{1}{2}\pi k)}{-k\pi} e^{-jk\pi/2} = \frac{2j}{k\pi}$

The integration of this is $a_k = \frac{2j}{k\pi} \cdot \frac{1}{jk\omega_0} = \frac{2}{k\pi \cdot 2\pi/4\, k}$
$= \frac{4}{k^2\pi^2}$, which is what is given above

+7

Signal: x_1

5/8

Part c. Determine the values of the Fourier series coefficients a_{-1} and a_1 for the signal $x_2(t)$ above.

$$a_k = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}$$

$T = 2$

$\Rightarrow \omega_0 = 2\pi/2 = \pi$

$$a_{-1} = \frac{1}{2}\int_{-1/2}^{3/2} [\delta(t) - 2\delta(t-1)]\, e^{-j\cdot -1\cdot\pi\cdot t}\, dt$$

$$= \frac{1}{2}\left[e^{j\pi t}\Big|_{t=0} - 2\, e^{j\pi t}\Big|_{t=1} \right]$$

$$= \frac{1}{2}[1 + 2\cdot 1] = \frac{1}{2}[+3] = +3/2$$

$$a_1 = \frac{1}{2}\int_{-1/2}^{3/2} [\delta(t) - 2\delta(t-1)]\, e^{-j\cdot 1\cdot\pi\cdot t}\, dt$$

$$= \frac{1}{2}\left[e^{-j\pi t}\Big|_{t=0} - 2e^{-j\pi t}\Big|_{t=1} \right]$$

$$= \frac{1}{2}[1 + 2\cdot 1] = \frac{1}{2}[1 + 2] = +3/2$$

$a_{-1} =$ $-1/2$ $\qquad a_1 =$ $-1/2$

PROBLEM 3 (20%)

Consider a *stable* DT system whose input-output relation is characterized by the following difference equation:

$$\frac{1}{2}y[n] - y[n-1] = x[n].$$

Part a. Find the unit sample response $h[n]$ of the system.

$x[n] = \delta[n]$

$h[n]$ has a particular component $A\,\delta[n]$

and the homogeneous components from

$\frac{1}{2}y[n] - y[n-1] = 0$. By inspection, the homogeneous

soln is $B \cdot 2^n$, because $\frac{1}{2}B \cdot 2^n - B\,2^{n-1} = 0$

Since the system is stable, and $2^n \to \infty$ as $n \to \infty$, our unit sample response has the form $h[n] = A\,\delta[n] + B \cdot 2^n\,u[-n]$.

To find the coefficients, evaluate the difference eqn at $t=0$ and $t=1$

At $t=0$, $\frac{1}{2}[A\,\delta[0] + B \cdot 2^0\,u[-0]] - [A\,\delta[-1] + B \cdot 2^{-1}\,u[1]] = \delta[0]$

$\Rightarrow \frac{1}{2}A + \frac{1}{2}B - B/2 = 1 \Rightarrow \frac{1}{2}A = 1 \Rightarrow A = 2$

At $t=1$, $\frac{1}{2}[A\,\delta[1] + B \cdot 2^1\,u[-1]] - [A\,\delta[0] + B \cdot 2^0\,u[0]] = \delta[1] = 0$

$\frac{1}{2}[0 + 0] - A + B = 0 \Rightarrow A = B \Rightarrow B = 2$

CLOSE

↓

+9

$h[n] =$ $2\,\delta[n] + 2 \cdot 2^n\,u[-n]$

Part b. Find the output $y_b[n]$ of the system when the input $x_b[n]$ is

$$x_b[n] = \cos\left(\frac{\pi}{3}n\right).$$

Write the input as an eigenfunction series

$$x_b[n] = \cos(\pi/3\, n) = \frac{1}{2} e^{j\pi/3\, n} + 1/2\, e^{-j\pi/3\, n}.$$

Next find the frequency response of the system by letting $y[n] = H(e^{j\omega}) e^{j\omega n}$ and $x[n] = e^{j\omega n}$:

$$\frac{1}{2} H(e^{j\omega}) e^{j\omega n} - H(e^{j\omega}) e^{j\omega(n-1)} = e^{j\omega n}$$

$$\frac{1}{2} H(e^{j\omega}) \cancel{e^{j\omega n}} - H(e^{j\omega}) \cancel{e^{j\omega n}} e^{-j\omega} = \cancel{e^{j\omega n}}$$

$$\frac{1}{2} H(e^{j\omega}) - H(e^{j\omega}) e^{-j\omega} = 1$$

$$H(e^{j\omega}) = \frac{1}{\frac{1}{2} - e^{-j\omega}}$$

$$H(e^{j\pi/3}) = \frac{1}{1/2 - e^{-j\pi/3}} = \frac{1}{1/2 - \cos(\pi/3) - j\sin(-\pi/3)} = \frac{1}{1/2 - \frac{1}{2} + j\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}j}$$

$$H(e^{-j\pi/3}) = \frac{1}{1/2 - e^{j\pi/3}} = \frac{1}{1/2 - \cos(\pi/3) - j\sin(\pi/3)} = \frac{1}{1/2 - 1/2 - \frac{\sqrt{3}}{2}j} = \frac{-2}{\sqrt{3}j}$$

$$y[n] = \frac{1}{2} H(e^{j\pi/3}) e^{j\pi/3\, n} + \frac{1}{2} H(e^{-j\pi/3}) e^{-j\pi/3\, n}$$

$$= \frac{1}{2}\left[\frac{2}{\sqrt{3}j} e^{j\pi/3\, n} - \frac{2}{\sqrt{3}j} e^{-j\pi/3\, n}\right]$$

$$= \frac{2}{\sqrt{3}} \sin(\pi/3\, n)$$

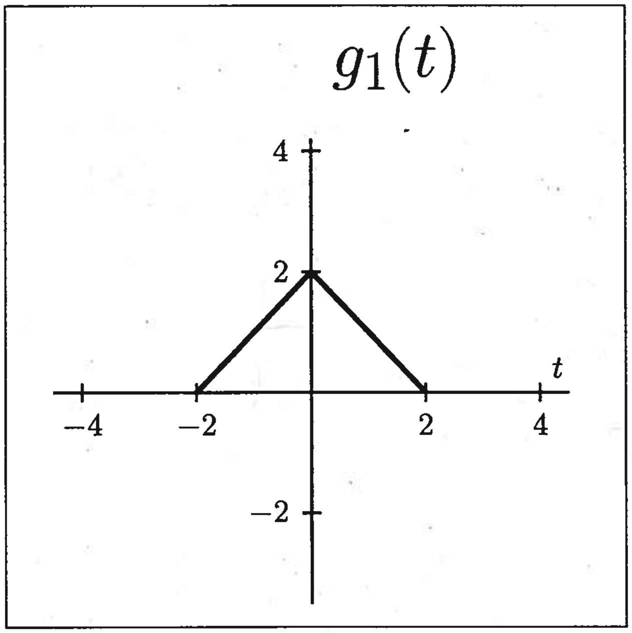
+10

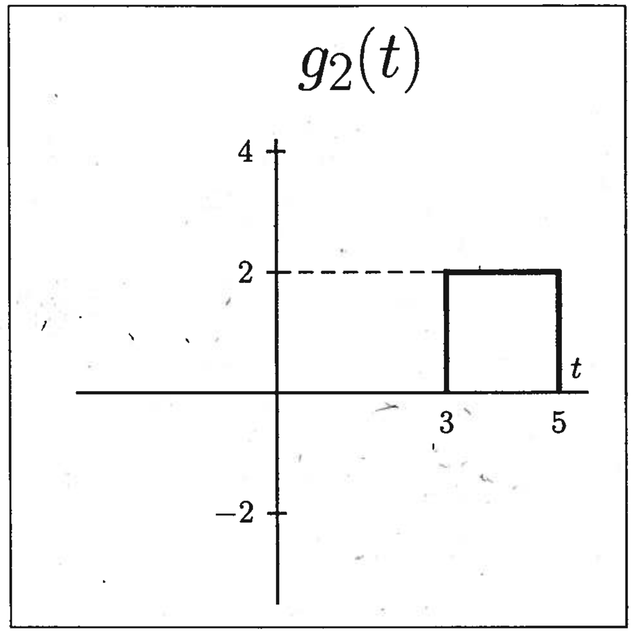
$y_b[n] =$ $2/\sqrt{3}\, \sin(\pi/3\, n)$

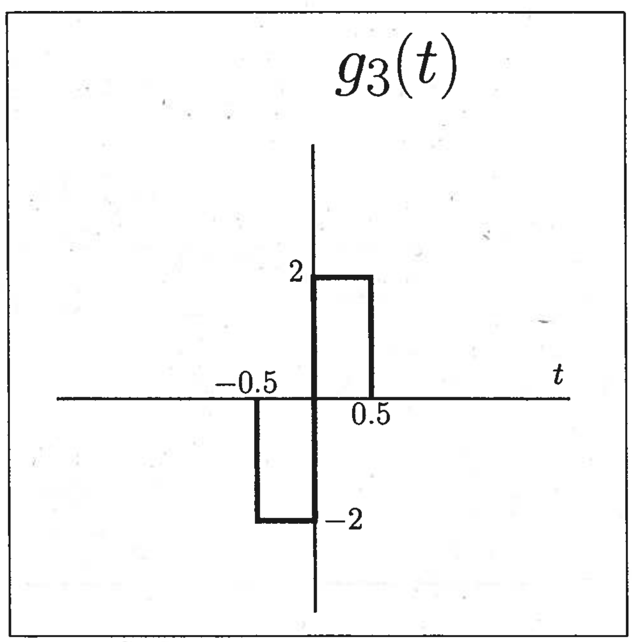
Note: $\cos\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$, $\cos\left(\frac{\pi}{3}\right) = \sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$

PROBLEM 4 (25%)

Consider the following three CT signals.







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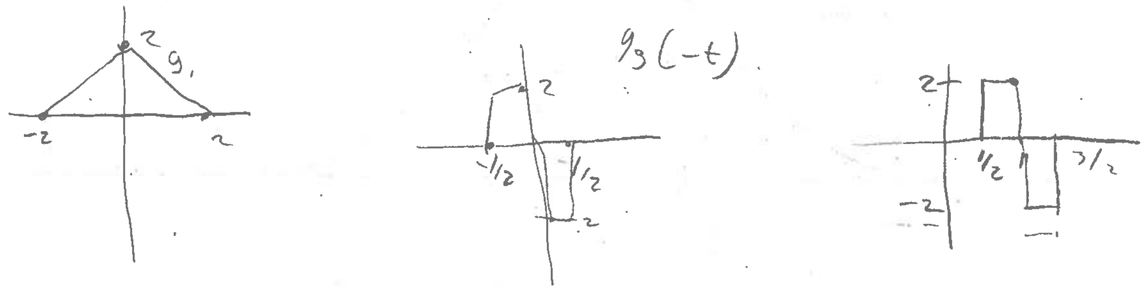
Part a. Suppose $y_a(t) = g_i(t) * g_j(t)$. What is the pair (i, j) with $i \neq j$, such that $y_a(1) = 1/2$?

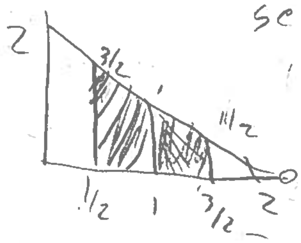
25/25

Check $g_1 * g_3$



$g_1 = 2 - t, \quad 0 \le t \le 2$

Graphically evaluating the convolution, this is twice the area of the first shaded section minus twice the area of the second.



$$2\left[\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2} - \frac{1}{2}\cdot 1\cdot 1\right]$$

$$-2\left[\frac{1}{2}\cdot 1\cdot 1 - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right]$$

$$= 2\left[\frac{9}{8} - \frac{1}{2}\right] - 2\left[\frac{1}{2} - \frac{1}{8}\right]$$

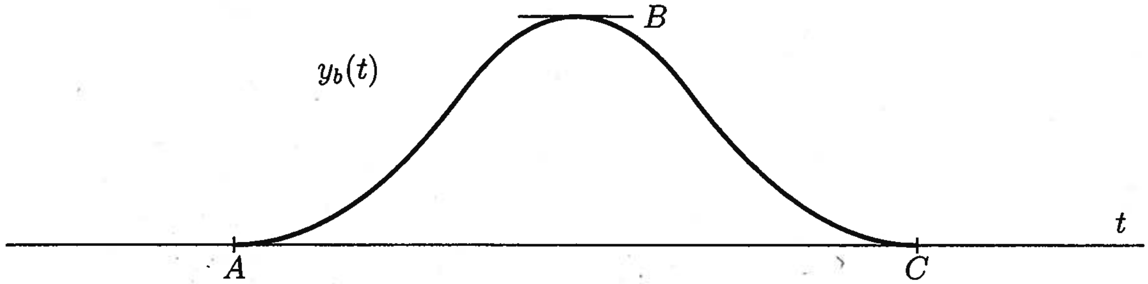
$$= 2\left[\frac{5}{8}\right] - 2\left[\frac{3}{8}\right]$$

$$= \frac{10}{8} - \frac{6}{8} = \frac{4}{8} = \frac{1}{2},$$ as expected.

+6

$i =$ 1 $\qquad j =$ 3 $\qquad$ note: $i \neq j$

Part b. Which pair (i, j) with $i \neq j$ of the above signals can be convolved to form $y_b(t) = g_i(t) * g_j(t)$ below, and what are values of A, B, and C depicted in the plot ?



Flip g_1 (no change) and slide along g_3

Overlap begins at $t = A = 1$.

maximum overlap at $t = 4$

At $t=4$ the value is twice the area of the shaded region:

$= 2\cdot\left[\frac{1}{2}\cdot 4\cdot 2 - \frac{1}{2}\cdot 1\cdot 1 - \frac{1}{2}\cdot 1\cdot 1\right] = 2[4-1] =$

$= 2\cdot 3 = 6 = B$

overlap ends at $t = C = 7$

+12

$i =$ 1 $\qquad j =$ 2 $\qquad$ note: $i \neq j$

$A =$ 1 $\qquad B =$ 6 $\qquad C =$ 7

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Part c. Which one of the three above signals is the impulse response for the CT system described by the following differential equation?

$$\frac{d}{dt}y(t) = -2\,x(t+0.5) + D\,x(t) - 2\,x(t-0.5).$$

What is the value of D ?

$\frac{d}{dt} y(t) = -2\,\delta(t+0.5) + D\,\delta(t) - 2\,\delta(t-0.5)$

$\Rightarrow$ $y(t)$ steps by $\begin{cases} -2 & \text{at } t=-0.5 \\ D & \text{at } t=0 \\ -2 & \text{at } t=0.5 \end{cases}$

This describes $g_3(t)$, with

$D = 4$ (by inspection)

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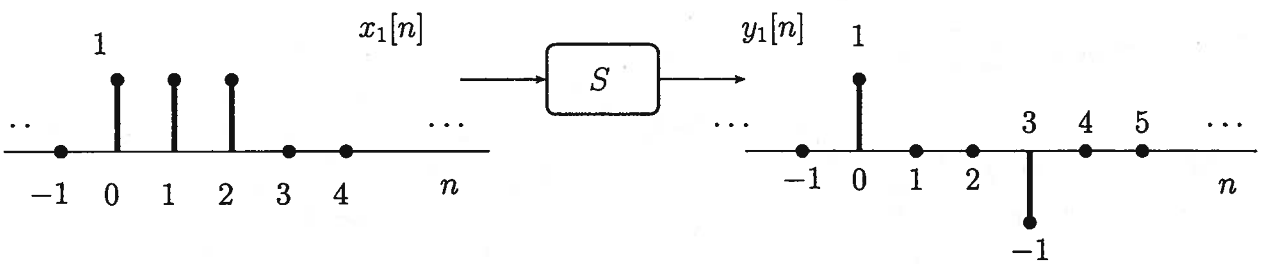
+7

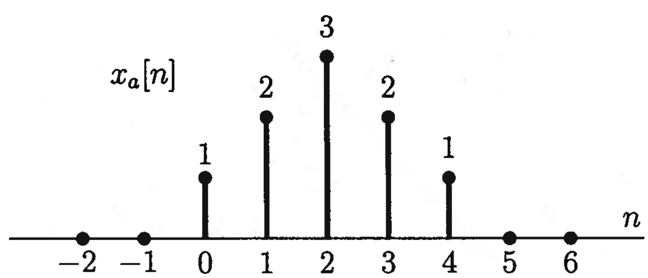
Signal: g_3 $D =$ 4

PROBLEM 5 (20%)

Consider a *causal* DT LTI system, S which produces the output $y_1[n]$ when the signal $x_1[n]$ is applied as input as shown below:



Part a. Plot the output $y_a[n]$ of the system S when the input $x_a[n]$ is the following. Clearly label the important numbers in the plot.



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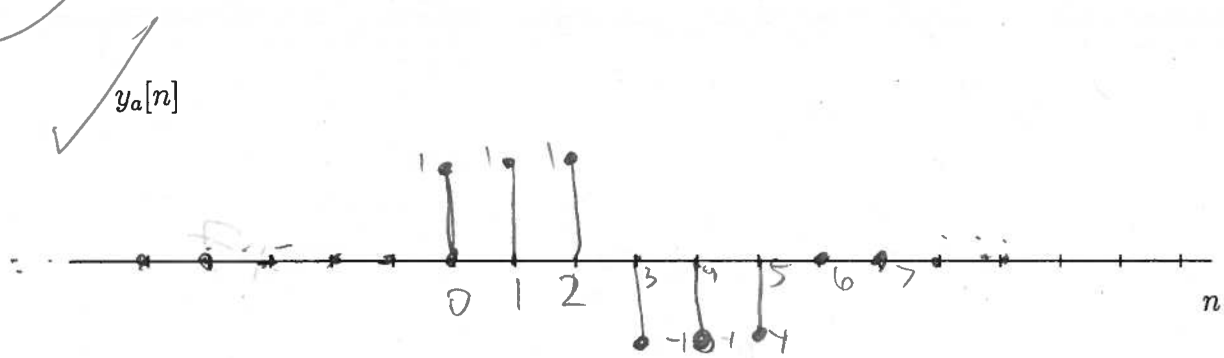

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Work Page for Problem 5

$x[n] = x_1[n] + x_1[n-1] + x_1[n-2]$

$\Rightarrow y[n] = y_1[n] + y_1[n-1] + y_2[n-2]$

 Problem 5 continues on the following page.

Part b. Graph the unit sample response $h[n]$ of the system S. Clearly label the important numbers in the plot.

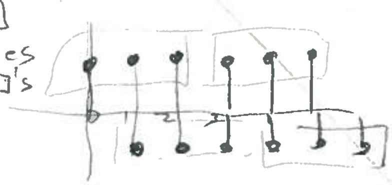
S is causal, so $h[n] = 0$, $n < 0$

$\delta[n]$ is an infinite series of $x_a[n]$'s:

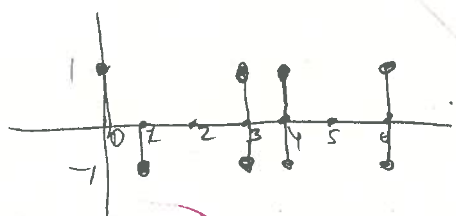
$\delta[n] = x_a[n] + x_a[n-1] + x_a[n-3] - x_a[n-4] \ldots$

So $h[n] = y_a[n] - y_a[n-1] + y_a[n-3] - y_a[n-4] \ldots$

$\delta[n]$ as a series of $x_a[n]$'s



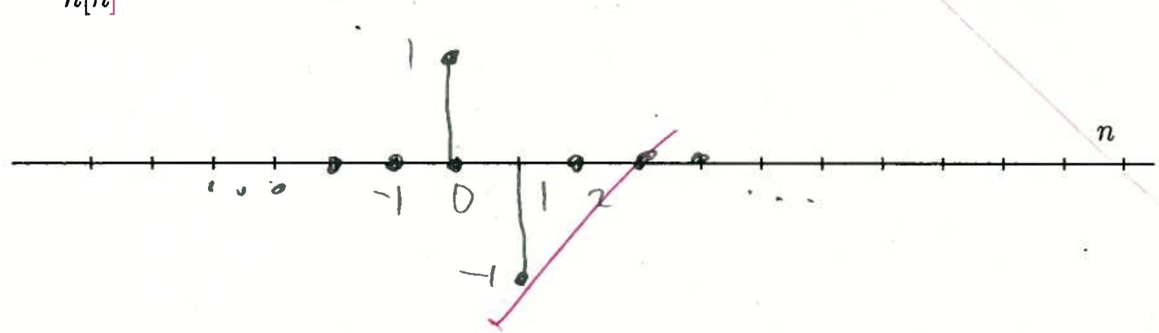
$h[n]$



all but the first two points cancel

10/10

$h[n]$



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Work Page