MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.003: Signals and Systems—Spring 2003

Quiz 2
Thursday, April 24, 2003

Directions: The exam consists of 4 problems on pages 1 to 13. Please make sure you have all the pages. Tables of transforms and properties are supplied to you at the end of this booklet. **Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. DO IT NOW!.** All sketches must be adequately labeled. Unless indicated otherwise, **answers must be derived or explained**, not just simply written down. This examination is closed book, but students may use two 8 1/2 × 11 sheets of paper for reference. Calculators may not be used.

NAME: DAN PORTS

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Check your section	Section	Time	Room	Rec. Instr.	TA
☐	1	10-11	36-112	Prof. Daniel	Mario
☐	2	11-12	36-112	Prof. Daniel	Mario
☐	3	12- 1	36-112	Prof. D. Freeman	Salil, Peter
☐	4	1- 2	36-112	Prof. D. Freeman	Salil, Peter
☐	5	10-11	36-114	Prof. W. Freeman	Stephen
☐	6	11-12	34-302	Prof. W. Freeman	Stephen
■	7	12- 1	34-304	Taka	Siddhartan, Ariel
☐	8	1- 2	34-304	Taka	Siddhartan, Ariel

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Problem	No. of points	Score	Grader
1	25	24	S.G
2	25	25	K.T
3	25	25	AS
4	25	25	SD
Total	100	99	SG.

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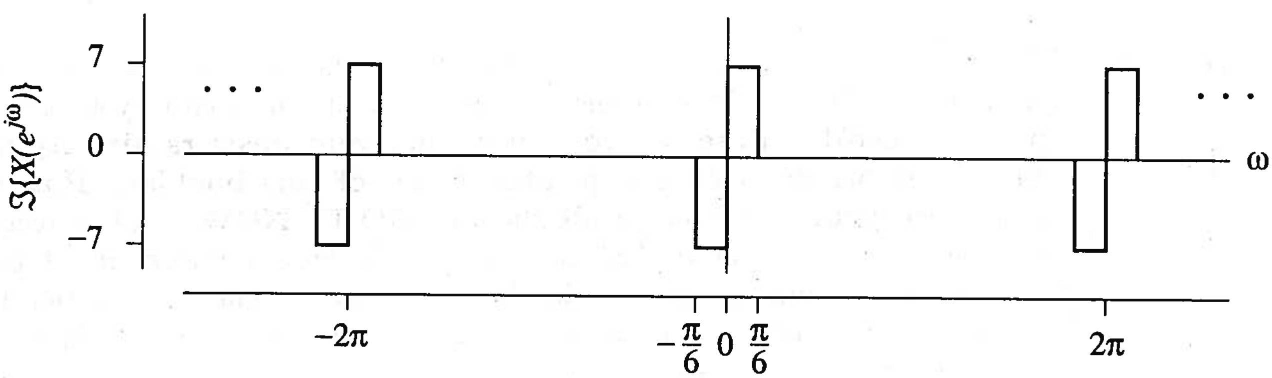
Problem 1 (25 points).
Let $x[n]$ represent a discrete time signal whose Fourier Transform $X(e^{j\omega})$ is given by

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

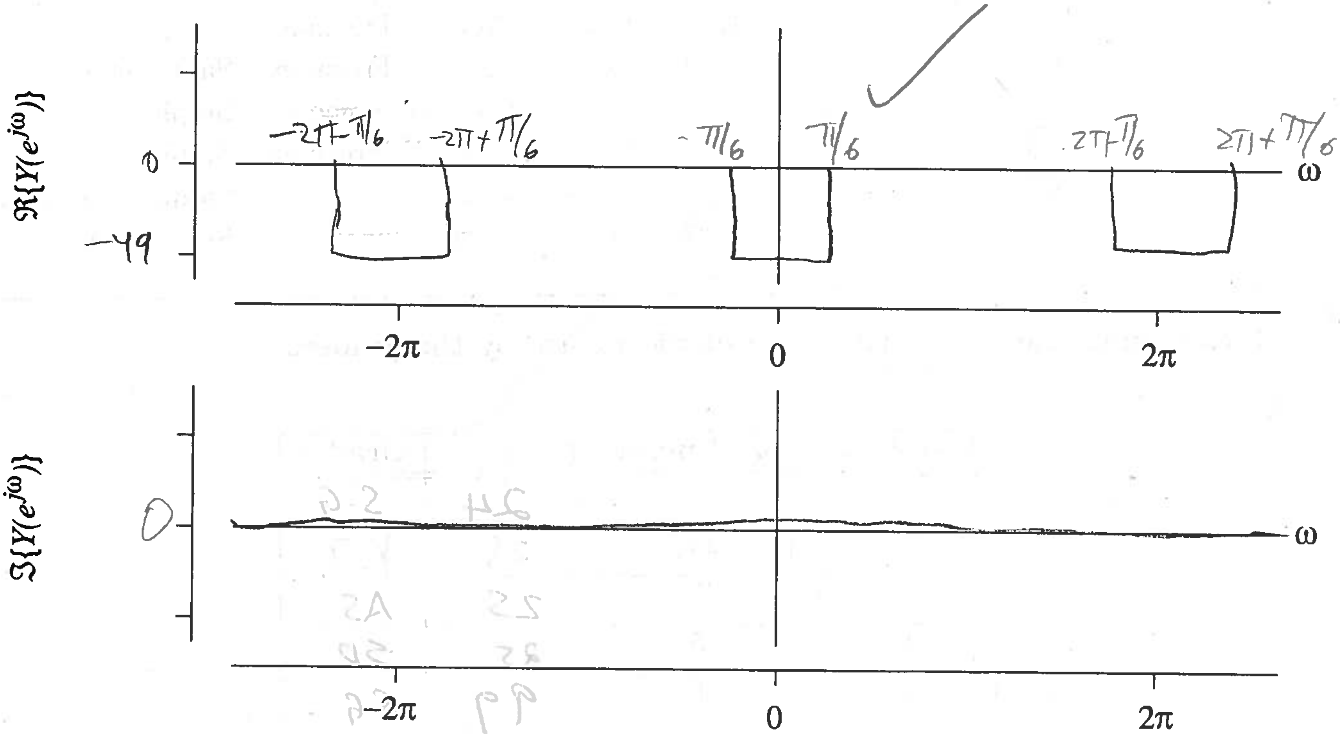
where the real part $\Re\{X(e^{j\omega})\}$ is zero and the imaginary part $\Im\{X(e^{j\omega})\}$ is given below.



Part 1a. Let $y[n]$ represent the convolution of $x[n]$ with $x[n]$, i.e., $y[n]= x[n] * x[n]$. Sketch the real and imaginary parts of $Y(e^{j\omega})$, the Fourier transform of $y[n]$ on the axes below. Label all axes.

+8

$x[n] \longrightarrow \boxed{x[n]} \longrightarrow y[n]$



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$y[n] = x[n] * x[n]$

$\Rightarrow Y(e^{jw}) = X(e^{jw}) \cdot X(e^{jw}) = (X(e^{jw}))^2$

$$X(e^{jw}) = \begin{cases} -7j & -\pi/6 < w < 0 \\ 7j & 0 < w < \pi/6 \end{cases}$$

(and periodic w/ $T = 2\pi$)

$$\Rightarrow X^2(e^{jw}) = \begin{cases} (-7j)^2 = (-1)^2(7)^2(j)^2 = -49 & -\pi/6 < w < 0 \\ (7j)^2 = (7^2)(j^2) = -49 & 0 < w < \pi/6 \end{cases}$$

(and periodic w/ period 2π)

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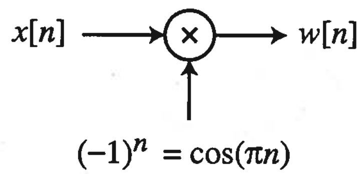
Part 1b. Let $w[n]$ represent the product of $x[n]$ with $(-1)^n$, i.e., $w[n]= (-1)^n \times x[n] = \cos(\pi n) \times x[n]$. Sketch the real and imaginary parts of $W(e^{j\omega})$, the Fourier transform of $w[n]$ on the axes below. Label all axes.



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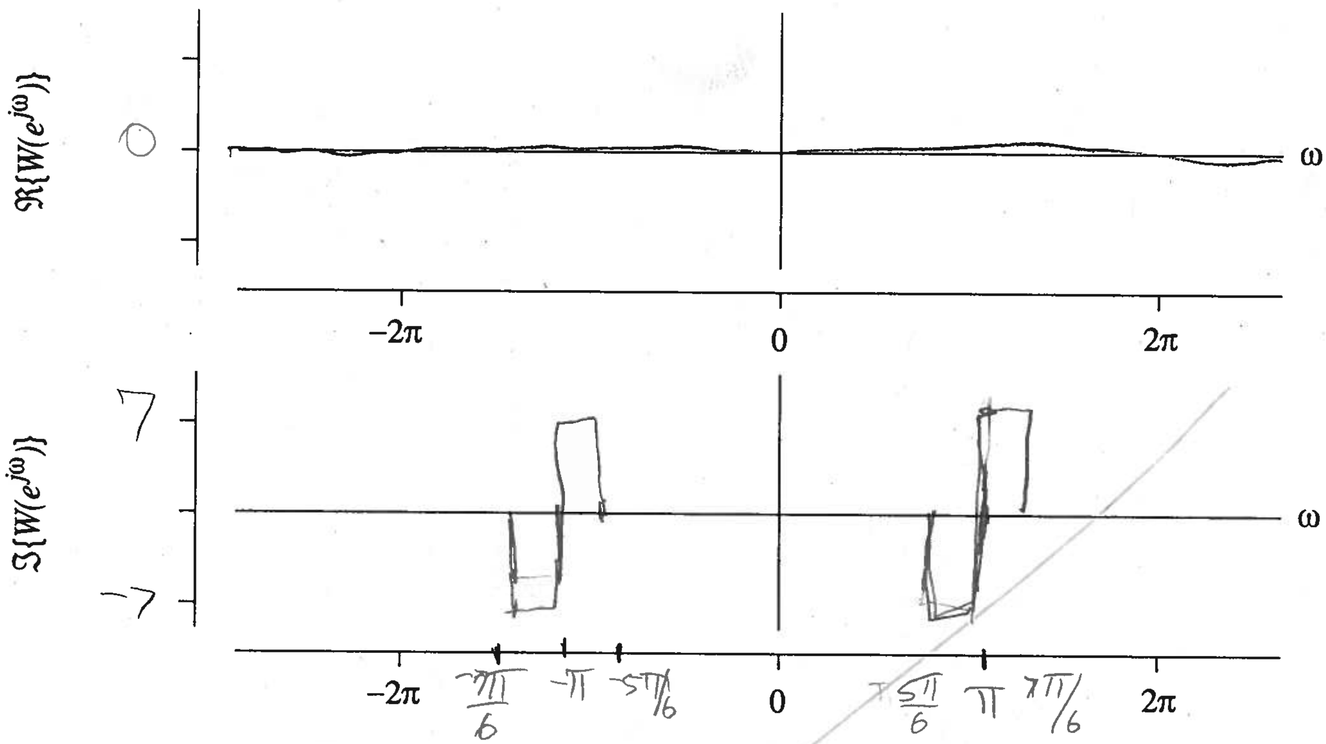

$(-1)^n = e^{j\pi n}$

$e^{j\pi n} x[n] \xrightarrow{\mathcal{F}} X(e^{j(\omega-\pi)})$

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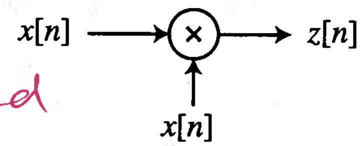
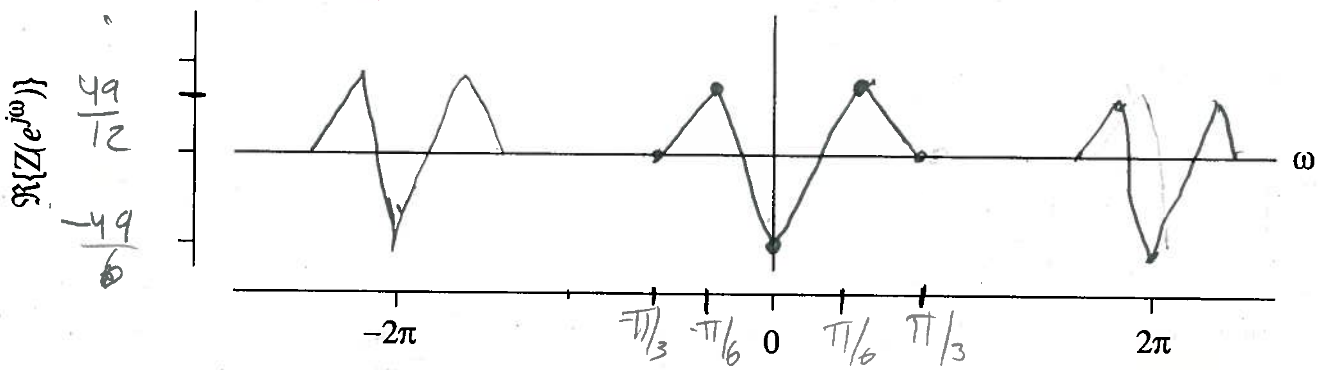
Part 1c. Let $z[n]$ represent the product of $x[n]$ with $x[n]$, i.e., $z[n] = x[n] \times x[n]$. Sketch the real and imaginary parts of $Z(e^{j\omega})$, the Fourier transform of $z[n]$ on the axes below. Label all axes.

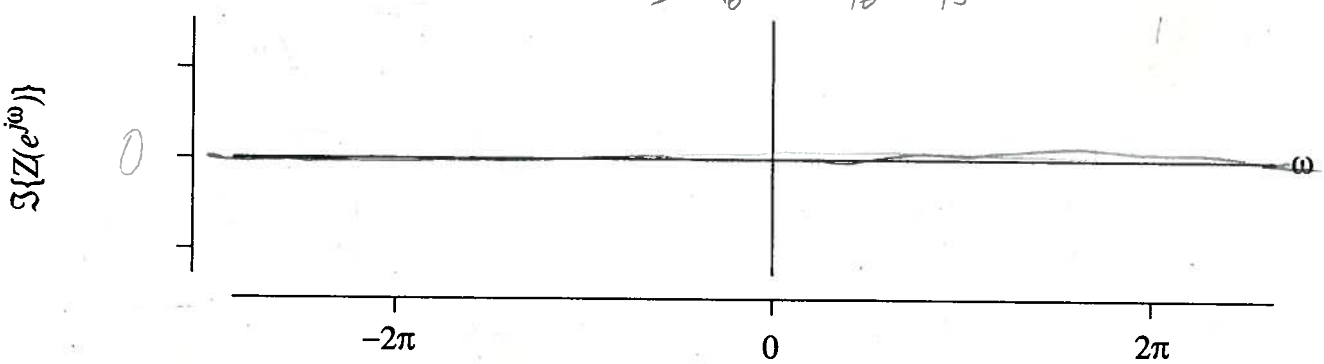
+8

Sign flipped −1



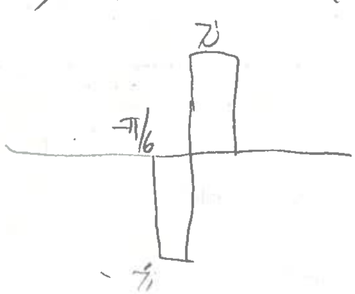
Periodic w/ $T = 2\pi$

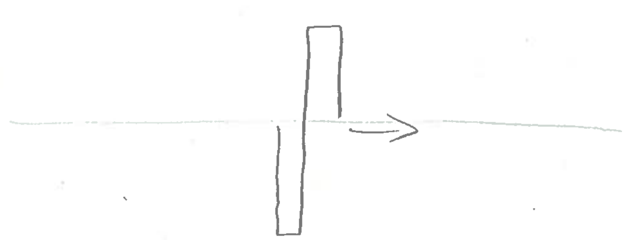




$Z(e^{jw}) = \frac{1}{2\pi} X(e^{jw}) \circledast X(e^{jw})$

Performing the convolution,





x[n] real / even

x[n]^2 real/even

$Z(e^{jw})$ conj. symm.

at $\pm\pi/3$ 0

at $\pm\pi/6$ $(7j \cdot -7j)\ \pi/6\ \frac{1}{2\pi}$

$= \frac{49}{12}$

at $t=0$ $7j \cdot 7j \cdot \pi/6 \cdot 1/2\pi \cdot 2$

$= -49\ \pi/6 \cdot 1/\pi$

$= -49/6$

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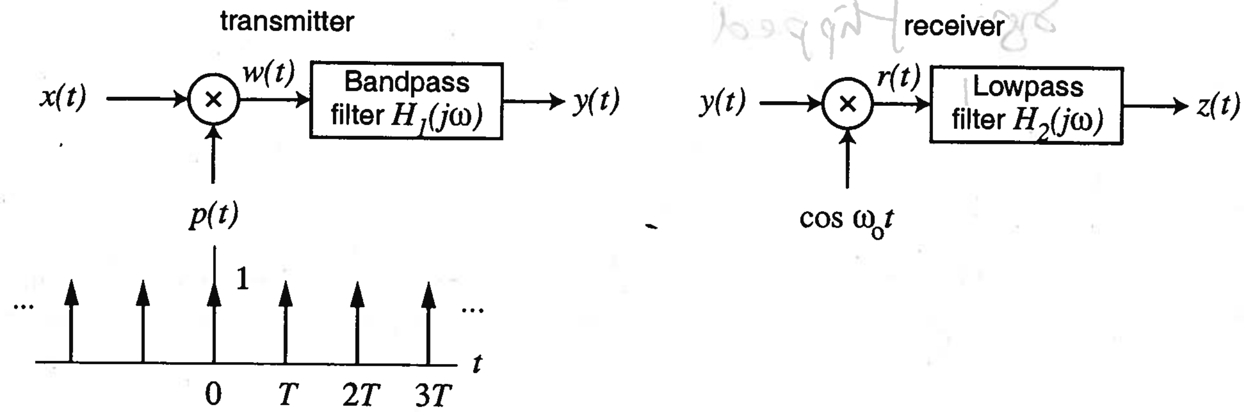
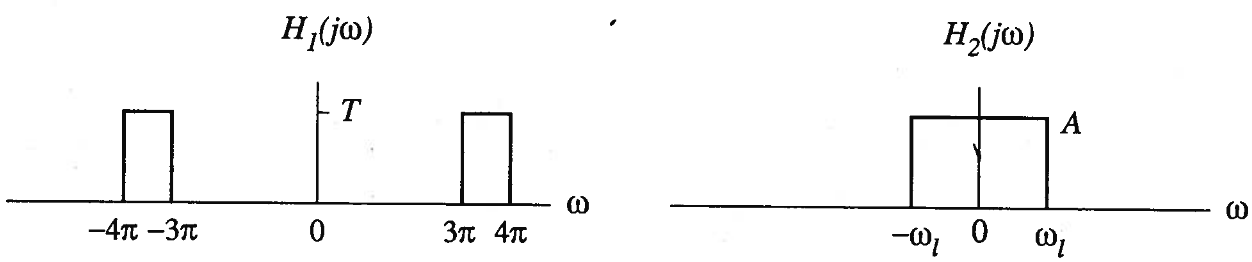
25/25

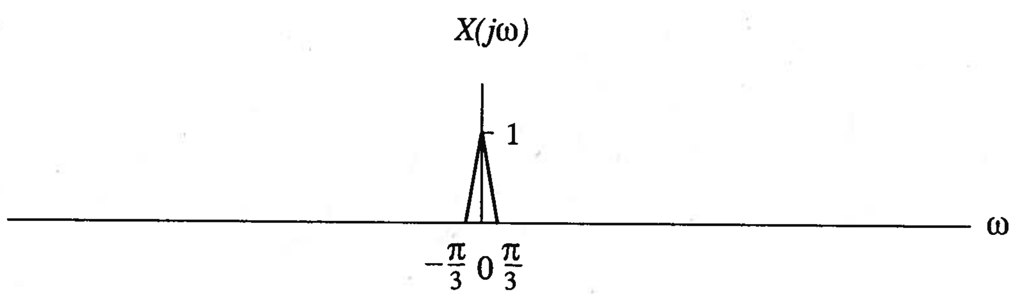
Problem 2 (25 points).
Consider the following communications system where $x(t)$ is the message, $y(t)$ is the signal that is transmitted and then received, and $z(t)$ is the decoded signal.



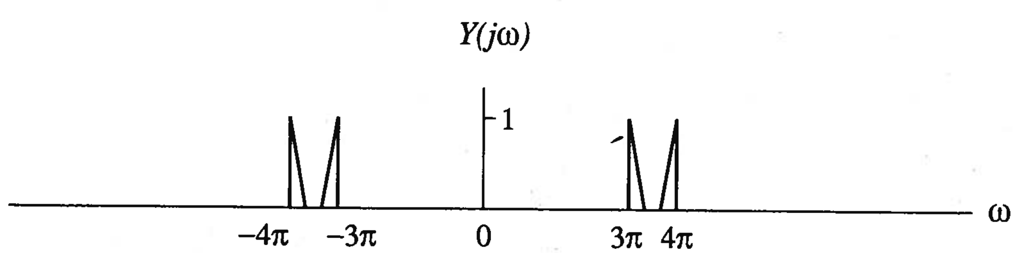
The filters are ideal with cutoff frequencies indicated below.



Assume $x(t)$ is bandlimited, with a Fourier transform given as follows.



Also assume that the Fourier transform of $y(t)$ is $Y(j\omega)$ as shown below.



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Part 2a. Determine the period T of $p(t)$.

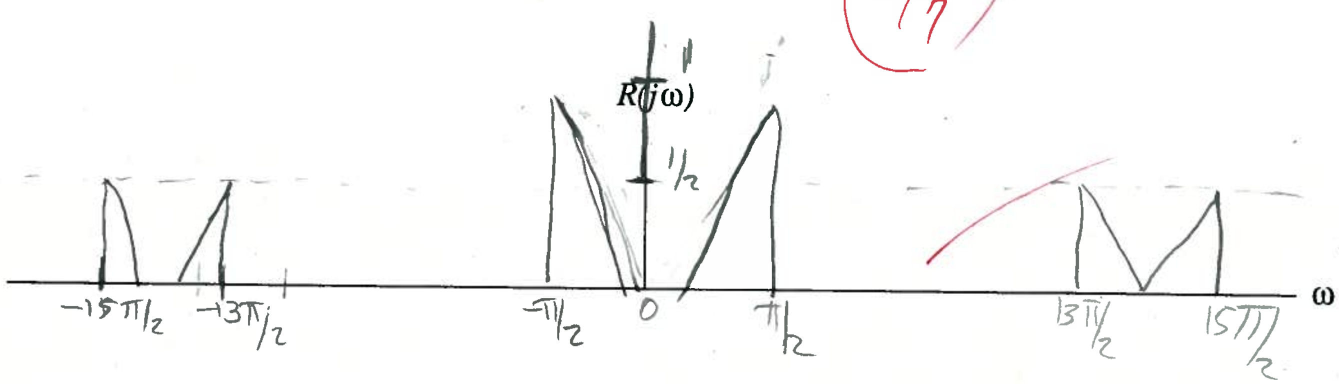
$T = \boxed{2}$

3/3

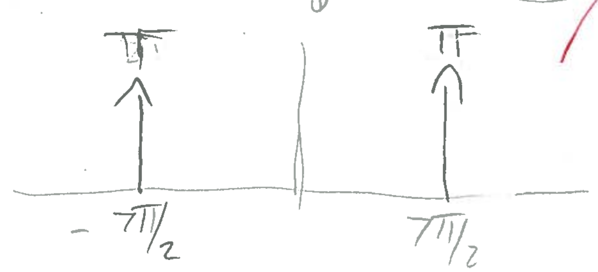
$W(j\omega)$ must have $X(j\omega)$ triangles centered at -4π, -3π, 3π, 4π to give the desired $Y(j\omega)$, so $P(j\omega)$ must have impulses spaced every π. Taking the FT of an impulse train, we must have $\frac{2\pi}{T} = \pi \Rightarrow T = 2$ to give the correct spacing in $P(j\omega)$.

Part 2b. Suppose $\omega_0 = 7\pi/2$. Sketch the resulting Fourier transform $R(j\omega)$ of $r(t)$. Fully label all axes.

7/7



Convolve $M(j\omega)$ w/ $\mathcal{F}\{\cos(\frac{7\pi}{2}t)\}$ and divide by 2π



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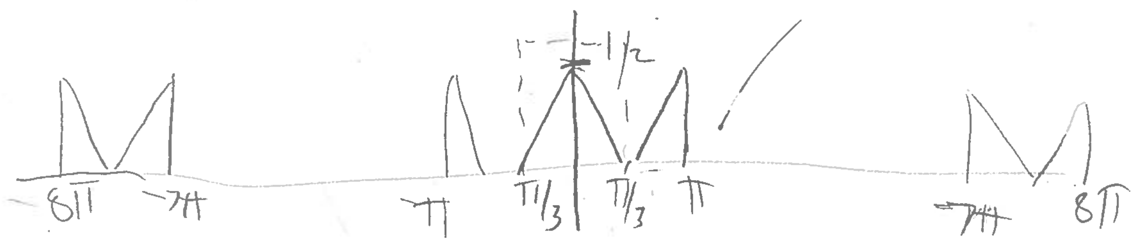
Part 2c. Determine values of ω_0 and ω_l and A so that $z(t)=x(t)$. [There may be more than one valid solution.]

1) 7/10

ω_0 = $\boxed{4\pi}$ ω_l = $\boxed{\pi/3}$ A = $\boxed{2}$

multiplying by $\cos(4\pi t)$ gives



We filter w/ gain 2 and cutoff freq $\pm \pi/3$ to recover $X(j\omega)$ (dotted line on graph)

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Problem 3 (25 points).
Consider the following three Laplace transforms with unspecified regions of convergence (ROC).

25/25

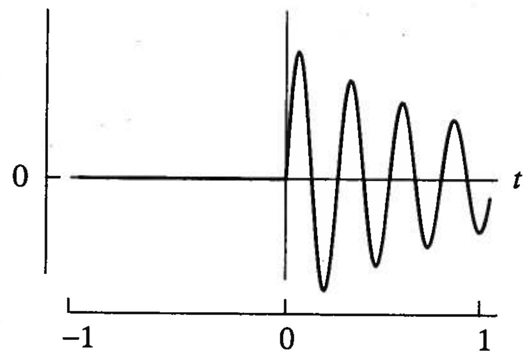
Transform 1: $\dfrac{1}{s^2+2s+626}$

Transform 2: $\dfrac{1}{s^2+2s-3}$

Transform 3: $\dfrac{1}{s^2+3s+2}$

Identify which transform was generated by each of the following time functions, and give the associated region of convergence.

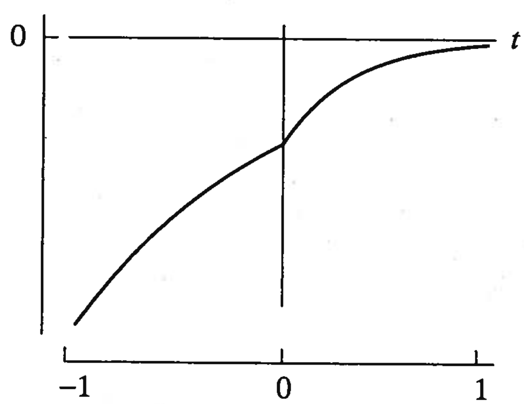
A.



Which transform: 1 ✓

Which ROC: $Re(s) > -1$ ✓

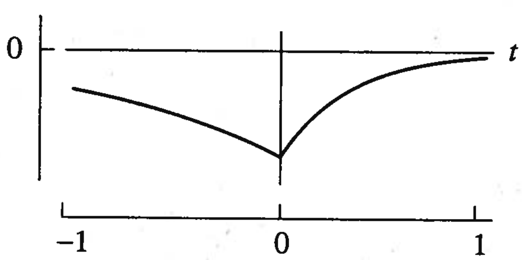
B.



Which transform: 3 ✓

Which ROC: $-2 < Re(s) < -1$ ✓

C.



Which transform: 2 ✓

Which ROC: $-1 < Re(s) < 3$ ✓

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T_1 $\dfrac{1}{s^2+2s+626} = \dfrac{1}{(s+1)^2+625}$

$= \dfrac{1}{625}\,\dfrac{(25^2)}{(s+1)^2+(25)^2}$

$\xrightarrow{\mathcal{L}^{-1}} e^{-t}\dfrac{\cos(25t)}{625}u(t)$ w/ ROC $Re(s) > -1$

This is fcn A.

T_2 $\dfrac{1}{s^2+2s-3} = \dfrac{1}{(s-3)(s+1)} = \dfrac{\frac{1}{(1-3)}}{s-3} + \dfrac{\frac{1}{-3-1}}{s+1}$

$= \dfrac{1/4}{s-3} + \dfrac{-1/4}{s+1}$

$\xrightarrow{\mathcal{L}^{-1}}$ $\dfrac{-e^{3t}u(-t) - e^{-t}u(t)}{4}$ when the ROC is $Re(s) < 3 \cap Re(s) > -1$

This is fcn C

T_3 $\dfrac{1}{s^2+3s+2} = \dfrac{1}{(s+1)(s+2)} = \dfrac{\frac{1}{(2-1)}}{s+1} + \dfrac{\frac{1}{(1-2)}}{s+2}$

$= \dfrac{1}{s+1} - \dfrac{1}{s+2}$

$\xrightarrow{\mathcal{L}^{-1}} -e^{-t}u(-t) - e^{-2t}u(t)$

when the ROC is $Re(s) < -1 \cap Re(s) > -2$

This is fcn B

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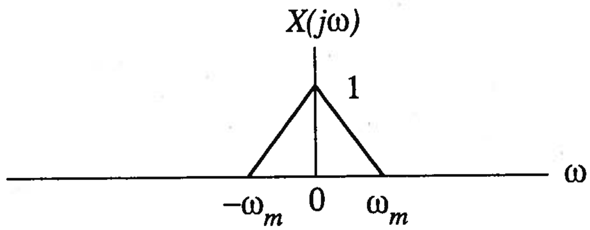
25/25

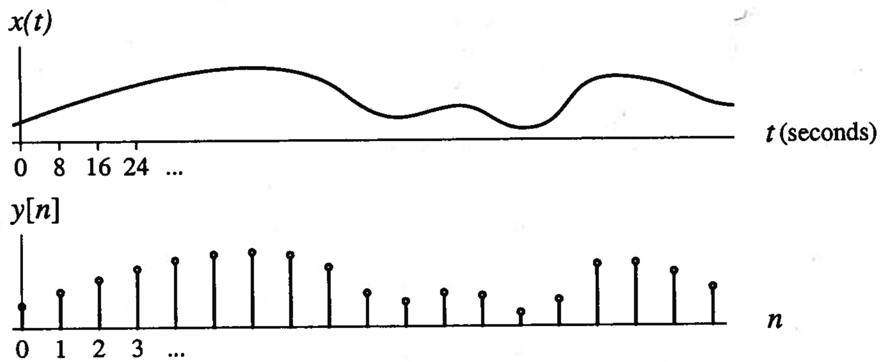
Problem 4 (25 points).
Consider a bandlimited, continuous time signal $x(t)$, whose Fourier Transform $X(j\omega)$ is given below.



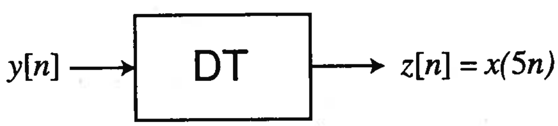
Sampling $x(t)$ with a sampling period of 8 seconds produces a discrete signal $y[n]$ such that

$$y[n] = x(8n).$$

The signals $x(t)$ and $y[n]$ are illustrated below.

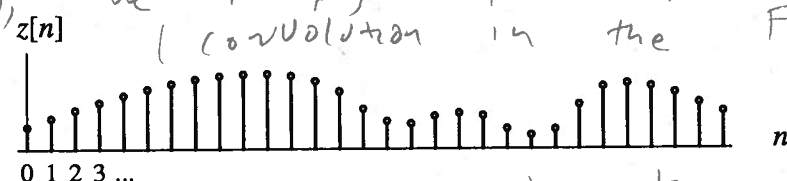


We wish to design a discrete time system that will take $y[n]$ as input and generate a discrete time signal $z[n]$ where $z[n] = x(5n)$.

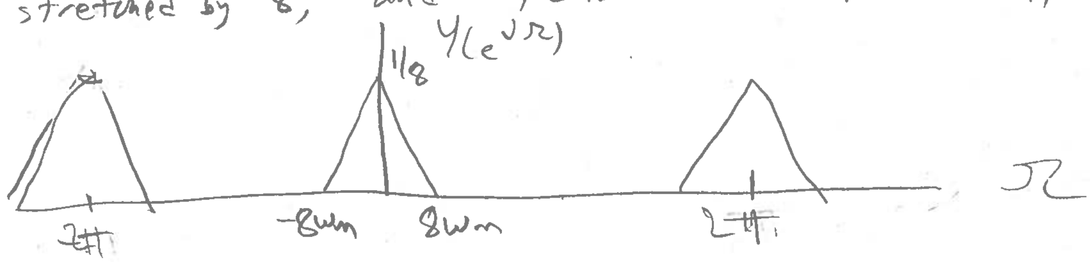


Notice that $z[n]$ is equal to the sequence that would have resulted if $x(t)$ had been sampled with a sampling period of 5 seconds.

To obtain $Y(e^{j\Omega})$, we multiply w/ an impulse train with period 8. (convolution in the Fourier domain)



Then we do a CT → DT conversion that scales the freq axis by $\Omega = \omega T = 8\omega$. This gives us the original FT, scaled by 1/8, stretched by 8, and periodic with period 2π



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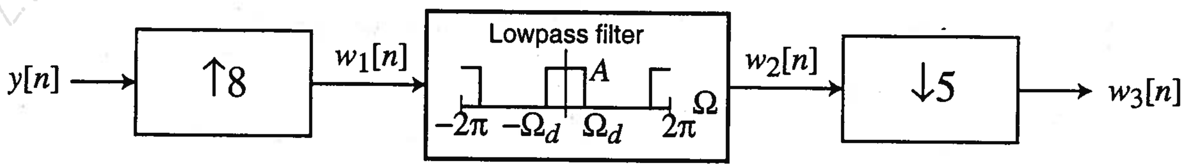
Suppose we implement the transformation as follows.

Let $y[n]$ represent the input to the discrete time system.
Let $w_1[n]$ represent the result of inserting 7 zeros between adjacent samples of $y[n]$ so that

$$w_1[n] = \begin{cases} y[n/8], & \text{if } n \text{ is an integer multiple of } 8 \\ 0, & \text{otherwise} \end{cases}$$

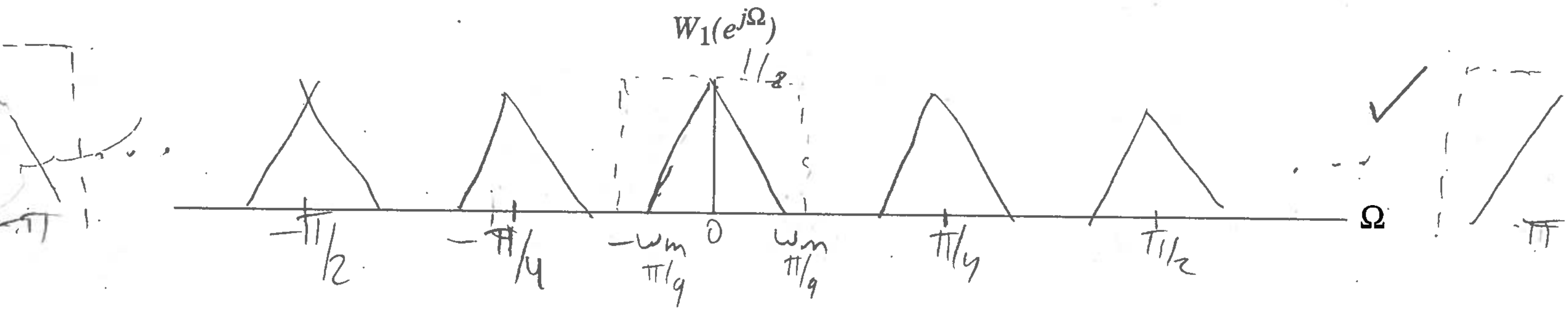
Let $w_2[n]$ represent the result of filtering $w_1[n]$ with a lowpass filter with a cutoff frequency Ω_d.
Let $w_3[n]$ represent the sequence containing every fifth sample in $w_2[n]$, i.e., $w_3[n] = w_2[5n]$.

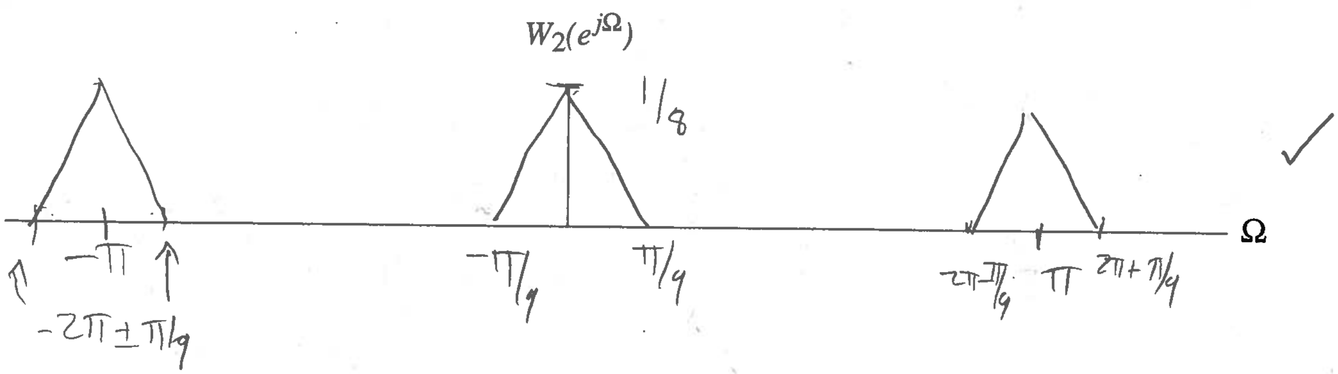


Part 4a. Plot $W_1(e^{j\Omega})$ and $W_2(e^{j\Omega})$, the Fourier transforms of $w_1[n]$ and $w_2[n]$, respectively, for the case when $\omega_m = \pi/9$ and $\Omega_d = \pi/8$ and $A = 1$. Fully label all axes.

$Y(e^{j\Omega})$ is sketched on the previous page.
$W_1(e^{j\Omega})$ is $Y(e^{j\Omega})$ shrunk! $W_1(e^{j\Omega}) = Y(e^{8j\Omega})$

$W_2(e^{j\Omega})$ is just $W_1(e^{j\Omega})$ LPFed (using the dashed lines on the $W_1(e^{j\Omega})$ plot).
Both are periodic w/ period 2π





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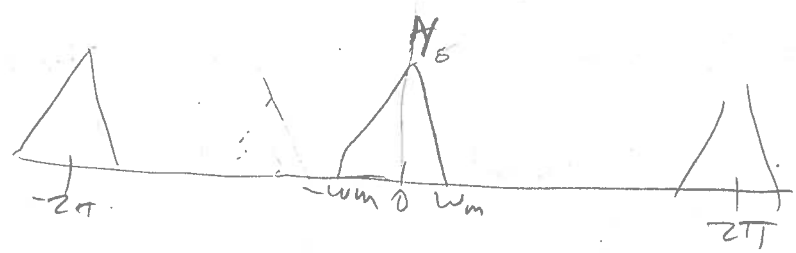
Part 4b. What are the maximum value of the bandlimit ω_m so that $w_3[n]=z[n]$?
What is the corresponding value of the cutoff frequency Ω_d and filter amplitude A?

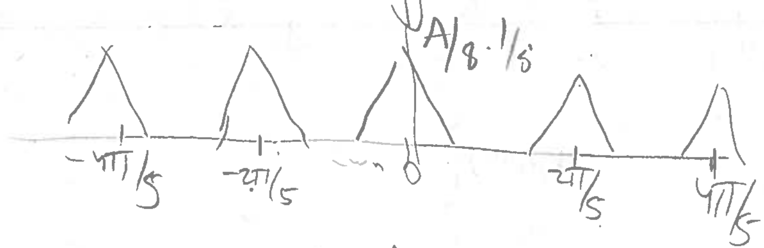
maximum value of ω_m = $\boxed{\pi/8}$ corresponding value of Ω_d = $\boxed{\pi/8}$

corresponding filter amplitude A = $\boxed{8}$

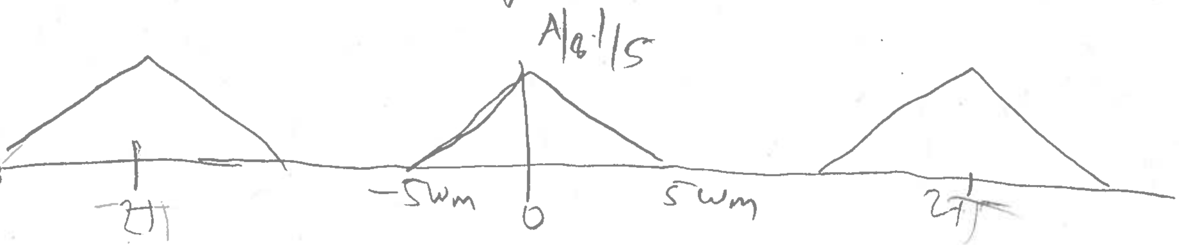
To downsample, we multiply by a unit sample train with period 5 and then decimate the zeros. This corresponds to convolving with an impulse train w/ height $2\pi/5$ and period $2\pi/5$, then stretching by 5 in the frequency domain!



convolve w/ impulse train



decimate (stretch)



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For complete recovery, we must prevent aliasing by ensuring no overlap.

For this process, that requires $\frac{2\pi}{5} - \omega_m > \omega_m$

$\Rightarrow 2\omega_m < \frac{2\pi}{5} \Rightarrow \omega_m < \pi/5$

However, to prevent aliasing when we sample $x(t) \to y[n]$ we must ensure no overlap between the triangles in the diagram drawn at the bottom of p. 11:

$2\pi - 8\omega_m < 8\omega_m \Rightarrow 16\omega_m < 2\pi \Rightarrow \omega_m < \pi/8$

This is the Nyquist criterion for the sampling

So the max freq is $\omega_m = \pi/8$

We must LPF w/ $\Omega_d = \omega_m = \pi/8$

We set $A = 8$ to undo the multiplication by $1/8$ inherent in the convolution w/ the impulse train involved in sampling $x(t)$.

The downsampling $w_2[n] \to w_3[n]$ doesn't change the amplitude.

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