Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Prof. Albert Meyer and Dr. Radhika Nagpal

Solutions cover sheet
November 15

Student's Solutions to Problem Set 11-12

Your name: Dan Ports

Due date: November 25

Submission date: 11/25

Circle your TA: Adrian George Josh Karen Lee Min Nikos Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	3
2	3
3	3
4	3
5	3
6	3
7	3
Total	21

[1] People other than course staff.

[2] Give citations to texts and material other than the class texts, handouts, and last term's bible.

Dan Portz
6.042 PS 11-12

1.a. If A and B are independent, we know that A and $\bar{B}$ are too. So
$Pr\{A|B\} = Pr\{A\}$
and $Pr\{A|\bar{B}\} = Pr\{A\}$.
Therefore, $Pr\{A|B\} = Pr\{A|\bar{B}\}$.

If $Pr(A|B) = Pr(A|\bar{B})$, then
$\frac{Pr(A \wedge B)}{Pr(B)} = \frac{Pr(A \wedge \bar{B})}{Pr\{\bar{B})}$ by def of conditional prob

$\Rightarrow (1 - Pr(B))\, Pr(A \wedge B) = Pr(B)\, Pr(A \wedge \bar{B})$
$\Rightarrow Pr(A \wedge B) = Pr(B)\,[Pr(A \wedge B) + Pr(A \wedge \bar{B})]$
$\Rightarrow \frac{Pr(A \wedge B)}{Pr(B)} = Pr(A)$ because B and $\bar{B}$ span the probability space
$\Rightarrow Pr(A|B) = Pr(A)$ by def of cond prob.
$\Rightarrow$ A and B are independent by def.

So $Pr\{A|B\} = Pr\{A|\bar{B}\} \Longleftrightarrow$ A, B independent.

b. By definition of mutual independence, A is independent of B, C, and $B \cap C$:
$Pr\{B|A\} = Pr(B)$
$Pr(C|A) = Pr(C)$
$Pr(B \cap C|A) = Pr(B \cap C)$
Therefore:
$Pr(B \cup C|A) = Pr(B|A) + Pr(C|A) - Pr(B \cap C|A)$
by inclusion/exclusion
$= Pr(B) + Pr(C) - Pr(B \cap C)$ from above
$= Pr(B \cup C)$ by inclusion/exclusion.

$Pr(B \cup C|A) = Pr(B \cup C)$, so $B \cup C$ and A are independent if A, B, C are mutually independent

2. a. 10% of problems have errors:

$Pr(E) = .10$ ✓

TA is correct 80% of time:

$Pr(T|E) = Pr(\bar{T}|\bar{E}) = .80$ ✓

Lecturer correct 75% of time:

$Pr(L|E) = Pr(\bar{L}|\bar{E}) = .75$ ✓

Lecturer and TA correctness are independent:

$Pr(L \wedge T|E) = Pr(L|E) \cdot Pr(T|E)$ ✓

$Pr(\bar{L} \wedge \bar{T}|\bar{E}) = Pr(\bar{L}|\bar{E}) \cdot Pr(\bar{T}|\bar{E})$ ✓

b. $Pr(L \wedge T) = Pr(L \wedge T|E)Pr(E) + Pr(L \wedge T|\bar{E})Pr(E)$

by total probability

$= Pr(L|E)Pr(T|E)Pr(E)$

$+ [1 - Pr(\bar{L}|\bar{E})][1 - Pr(\bar{T}|\bar{E})](Pr(\bar{E})$

$= (3/4)(4/5)(1/10) + (1/4)(1/5)(9/10)$

$= 12/200 + 9/200 = \boxed{21/200}$ ✓

c. These events are not independent.

If they were, $Pr(T)Pr(L)$ would equal $Pr(T \wedge L)$.

$Pr(T) = Pr(T|E)Pr(E) + Pr(T|\bar{E})Pr(E)$ (total prob)

$= Pr(T|E)Pr(E) + (1 - Pr(\bar{T}|\bar{E}))Pr(E)$

(complement rule)

$= (4/5)(1/10) + (1 - 4/5)(9/10) = 4/50 + 9/50 = 13/50$

By the same reasoning, $Pr(L)$ is given by

$Pr(L) = (3/4)(1/10) + (1 - 3/4)(9/10) = 3/40 + 9/40 = 12/40$

Thus $Pr(L)Pr(T) = 13/50 \cdot 12/40 = 156/2000$

This does not equal $Pr(L \wedge T) = 21/200$, so ✓

L and T cannot be independent.

3. a. The player can only move on the ith turn if he has rolled doubles on all previous turns. If he rolls doubles on the third turn, we must take away the squares he advanced on each of his three rolls.

✓ $R = R_1 + R_2 I_1 + R_3 I_1 I_2 - (R_1 + R_2 + R_3) I_1 I_2 I_3$

b. Assume R_i is independent of R_j and I_i independent of I_j for any $i \neq j$.

Then, $E(R_i) = 2 \cdot 1/6 (1+2+3+4+5+6) = 7$

✓ and $E(I_i) = 1/6$ because it's an indicator and $Pr(I_i = 1) = 1/6$

Using linearity of expectation,

✓ $E(R) = 7 + 7/6 + 7/6^2 - 21/6^3$

$\boxed{= 1785/216 \approx 8.264}$

4. The probability of success on any try will be $(1/10)^4$ since a 9 must be chosen 4 times from 10 available digits. This means the expected number of tries before success is $1/(1/10)^4 = 10^4$.

The probability of a try having length i is $(1/10)^{i-1}(9/10)$ for $1 \leq i \leq 3$, because $i-1$ 9s must be chosen followed by a non-9. For $i=4$, it is simply $(1/10)^3$ because, after 3 9s are chosen, the next will end the try as a success if it is a 9 or as a failure otherwise. So the expected length is $\sum_{i=1}^{4} i \cdot Pr(\text{length } i) = \frac{9}{10} + 2 \cdot \frac{1}{10} \cdot \frac{9}{10} + 3 \cdot \frac{1}{10^2} \frac{9}{10} + 4 \cdot \frac{1}{10^3}$

$= 1111/1000$

By Wald's theorem, the expected number of digits is the expected number of tries multiplied by the expected number of digits per try. $10^4 \cdot \frac{1111}{1000} = \boxed{11110}$ ✓

5. Consider an ordered set of k numbers. There are k! permutations in total, and $(k-1)!$ if the greatest element is in the last position. This means that in a permutation of n elements, the probability that the kth element is greater than all preceding elements is $\frac{(k-1)!}{k!} = 1/k$. Define an indicator variable I_k to represent this. $I_k = 1$ w/ $1/k$ probability. The expected total number of digits greater than all preceding ones is, by linearity of expectation, the sum of the indicator variables expected values: $\sum_{k=1}^{n} E(I_k) = \boxed{\sum_{k=1}^{n} 1/k} = H_n$

6a. Let the indicator variable I_i equal 1 when the student w/ ith rank in 6.003 has a higher rank in 6.042, and 0 otherwise. $Pr\{I_i = 1\} = (i-1)/n$ because the student must have a rank no greater than $i-1$, out of n possible ranks. So the expected value of I_i is $(i-1)/n$. The expected number of students w/ higher 6.042 rank is, by linearity, the sum of these expected values: $\sum_{i=1}^{n} (i-1)/n = \frac{1}{n}\left(\frac{n(n+1)}{2} - n\right) = \boxed{\frac{n-1}{2}}$ ✓

6b. Apply the same reasoning: I_i indicates that the ith-ranked student in 6.003 has a k-higher ranking in 6.042.

$E(I_i) = \Pr\{I_i = 1\} = 0$ for $i < k$

$E(I_i) = \Pr\{I_i = 1\} = \frac{n-k}{n}$ otherwise

By linearity, the expected number of students is the sum of these:

$$\sum_{i=1}^{n} E(I_i) = \sum_{i=k}^{n} \frac{n-k}{n} = \boxed{\frac{1}{n}\left(\frac{(n+k)(n-k+1)}{2} - k(n-k+1)\right)}$$

✓

7.a. If the previous flip was tails, then we will flip the coin once. If it gives heads (probability 1/2), then it will be flipped F_H times. If it gives tails, then it can expect to be flipped F_T times.

$F_T = 1 + \frac{1}{2} F_H + \frac{1}{2} F_T$ ✓

Similarly, if the previous flip was heads, then we will flip it at least once more. If it gives heads, then we have HH and we are done, otherwise we are back to the F_T case:

$F_H = 1 + \frac{1}{2} \cdot 0 + \frac{1}{2} F_T$ ✓

The starting state is equivalent to beginning with a tail, so the expected number of flips from ~~the~~ start state $F_S = F_T$

$F_S = F_T = 1 + \frac{1}{2} F_H + \frac{1}{2} F_S = 1 + \frac{1}{2}(1 + \frac{1}{2} F_S) + \frac{1}{2} F_S$

$\Rightarrow F_S(1 - \frac{1}{2} - \frac{1}{4}) = 1 + \frac{1}{2}$

$\Rightarrow F_S(\frac{1}{4}) = \frac{3}{2} \Rightarrow \boxed{F_S = 6}$ ✓

b. Apply the same reasoning. Now F_T still equals $1 + \frac{1}{2} F_H + \frac{1}{2} F_T$, but $F_H = 1 + \frac{1}{2} \cdot F_H + \frac{1}{2} \cdot 0$, since a tails after heads ends the game and a heads repeats F_H. The starting state is still equivalent to having tails previously.

So $F_S = F_T = 1 + \frac{1}{2} F_H + \frac{1}{2} F_T$

$F_H = 1 + \frac{1}{2} F_H \Rightarrow F_H = 2$

$\Rightarrow F_S = 1 + \frac{1}{2} \cdot 2 + \frac{1}{2} F_S$

$\Rightarrow F_S(1 - \frac{1}{2}) = 1 + 1 \Rightarrow \boxed{F_S = 4}$

c. Let the starting probability of getting HT first be p, and the probability of getting HT first after getting heads previously be q.

Then $p = \frac{1}{2} q + \frac{1}{2} p$ because flipping tails returns to the same state.

$q = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0$, since after getting one head we will either have HH or HT with equal probability.

$\Rightarrow q = \frac{1}{2}$

$\Rightarrow p = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} p \Rightarrow p(1 - \frac{1}{2}) = \frac{1}{4} \Rightarrow p = \frac{1}{2}$

The probability of winning is $\frac{1}{2}$, so the game is fair and should have even odds.

1:1 odds