Student's Solutions to Problem Set 3

Your name: DAN PORTS

Due date: September 25

Submission date: 9/25

Circle your TA: Adrian George Josh Karen Lee Min Nikos Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	3
2	3
3	3
4	3
5	2
Total	14

[1]People other than course staff.

[2]Give citations to texts and material other than the class texts, handouts, and last term's bible.

Dan Ports
6.042 PS3

1.a. 5 equivalence classes

(0,0)

(0,1) (1,0)

(0,2) (1,1) (2,0)

(1,2) (2,1)

(2,2)

Explain how you created these.

b. $\forall a \forall b\ (a,b) \sim (a,b) \Leftrightarrow a+b=b+a$

$\therefore \sim$ is reflexive ✓

$(a,b) \sim (c,d) \rightarrow (c,d) \sim (a,b)$

$\Leftrightarrow a+d=b+c \rightarrow c+b=d+a$

$\therefore \sim$ is symmetric ✓

$(a,b) \sim (c,d) \wedge (c,d) \sim (e,f) \Rightarrow (a,b) \sim (e,f)$

$\Leftrightarrow a+d=b+c \wedge c+f=d+e \rightarrow a+f=e+b$

$\Leftrightarrow a-e=b-f \Rightarrow a+f=e+b$

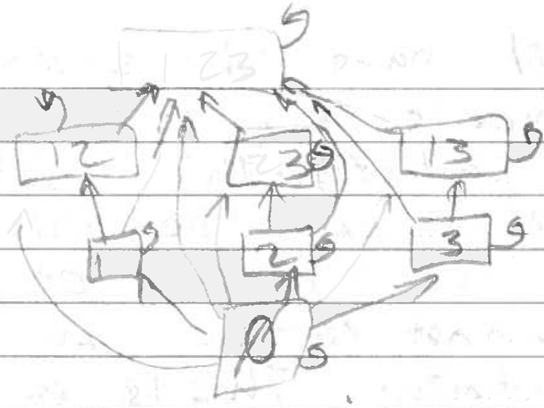
$\Leftrightarrow a+f=b+e \Rightarrow a+f=e+b$

$\therefore \sim$ is transitive ✓

$\sim$ is reflexive, symmetric, and transitive,
so $\sim$ is an equivalence relation ✓

c. There are $2n+1$ equivalence classes
because every number from 0 to $2n$
can be formed by adding pairs (from
$0+0$ to $n+n$). This creates $2n+1$
groupings into equivalence classes ✓

2a.



✓ $\emptyset \to 1 \to 12 \to 123$ and $\emptyset \to 2 \to 23 \to 123$ are maximal chains

✓ (12; 23; 13) and (1, 2, 3) are maximal antichains

∅, 1, 2, 3, 12, 23, 13, 123 is a topological sorting.

b. Consider $n=1$. Then the set is

✓ $\{1\}$, and the power set is $\{\emptyset, 1\}$. Clearly a 2-element chain exists: ∅ and 1. Assume that a chain of length $n+1$ exists for $A = \{1, 2, 3, \ldots, n\}$. Add a new element $n+1$. Then a new, longer chain can be formed by creating a new element by adding the integer $n+1$ to the final element of the chain. This satisfies the subset relation and creates a chain of length $n+2$. By induction, any set $A = \{1, 2, 3, \ldots, n\}$ has $n+1$ elements in a chain of $(\mathcal{P}(A), \subseteq)$

c. Assume the opposite. Then there exists a set M and a set N such that $M \subseteq A$, $N \subseteq A$, $|M| = |N| = k$ (by the definition of set B, since $M \in B \wedge N \in B$). Also, $M \subseteq N$ because we are assuming the set is not an antichain.

Since M and N have the same number of elements k, they must be equal to satisfy the subset relation. But M and N cannot be equal, because they are distinct elements of the set A, which cannot contain multiple identical elements by definition. This is a contradiction,

✓ so the assumption must be false: there are no distinct elements M and N, and B must therefore be an antichain.

3 a. R is reflexive: aRa is trivial because $a_n = a_n$

R is transitive: $aRb \wedge bRc \Rightarrow aRc$ because aRb implies $\exists m$ such that for all $x < m$ $a_x = b_x$ and $a_m < b_m$, and bRc implies $\exists n$ such that for all $y < n$ $b_y = c_y$ and $b_n < c_n$. Either $m < n$, in which case $\forall y < n$ $a_y = c_y$ and $a_n < c_n \Rightarrow aRc$, or $m > n$ in which case $\forall x < m$ $a_x = c_x$ and $a_m < c_m \Rightarrow aRc$. Thus $aRb \wedge bRc$ always implies aRc

R is antisymmetric: if aRb, then $\neg bRa$ because $\exists n$ such that $\forall x < n$ $a_x = b_x$ and $a_n < b_n$. It cannot also be the case that $a_n > b_n$.

Therefore R is a partial order. It is a total order because every pair of elements can be compared. Proof: assume two arbitrary strings A and B. If they have a prefix in common, remove it. Then there are three cases: neither string has any characters remaining, in which case they are lexicographically equal; one has characters remaining and the other does not, in which case the shorter one is lexicographically smaller; or they both have at least one character remaining in which it can be compared. This works for all strings since the alphabet is totally ordered and no assumptions were made in selecting the arbitrary strings A and B. Thus R is a total ordering.

b. R is not a well-ordering: the set

$\{b, ab, aab, aaab, \ldots\}$

has no least element under R

R^{-1} is not a well-ordering:

$\{a, aa, aaa, \ldots\}$ has no least element.

4. By induction: assume R^n is symmetric. Then show that R^{n+1} is symmetric:

$aR^{n+1}b \Rightarrow a R \circ R^n b$

$\Rightarrow \exists c \;\; aRc \wedge cR^n b$ by defn of composition

$\Rightarrow cRa \wedge bR^n c$ because R and R^n are symmetric

$\Rightarrow bR^n cRa$

$\Rightarrow b R^n \circ R a$ by defn of composition

$\Rightarrow b R^{n+1} a$

The base case R is symmetric by definition, so R^n is symmetric for all n

$R^* = \bigcup_{n=1}^{\infty} R^n$. R_n is symmetric for all n, so R^* must be symmetric because it is a union of symmetric sets

5.a. R_f satisfies the properties of an equivalence relation:

✓

R_f is reflexive

$a R_f a \Rightarrow f(a) = f(a)$ T

R_f is symmetric

$a R_f b \Rightarrow b R_f a$

$\Rightarrow f(a) = f(b) \Rightarrow f(b) = f(a)$ T

R_f is transitive

$a R_f b \wedge b R_f c \Rightarrow a R_f c$

$\Rightarrow f(a) = f(b) \wedge f(b) = f(c) \Rightarrow f(a) = f(c)$ T

Therefore R_f is an equivalence relation.

b Every equivalence relation E partitions its set A into a number of disjoint equivalence classes. For each equivalence class, designate a representative element. Then the function $f(a)$ that returns the representative element of a's equivalence class is a function such that $E = R_f$

Good, but Be explicit here: Show $a_1 R_f a_2 \leftrightarrow a_1 E a_2$