Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

Solutions cover sheet
October 11

Student's Solutions to Problem Set 6-7

Your name: Dan Ports

Due date: October 21

Submission date: 10/21

Circle your TA: Adrian George Josh Karen Lee Min Nikos Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	3
2	3
3	3
4	3
5	3
6	3
7	3
Total	21

[1]People other than course staff.

[2]Give citations to texts and material other than the class texts, handouts, and last term's bible.

Dan Portz
6.042 ps 6

1a. Proof: by structural induction, using the set of four statements as the induction hypothesis:

The tree can be constructed in two ways. First, it can be one node v only. Then it has maxval(T) = minval(T) = value(v). In this case the four statements of the induction hypothesis trivially hold, because there is only one move, and only one "strategy" with payoff equal to both the max and min value.

Next generate a new tree T from a set S of subtrees. Assume the induction hypothesis holds for all elements of S. Then each of the four statements holds for T.

1) max-val(T) is the maximum of the min-vals of the elements of S. The max-player can choose the move leading to this value. By part 2 of the IH, he is then guaranteed a payoff of at least min-val(T') = max-val(T)

2) If the max-player moves second he is choosing a move from one of the trees in S. Call this tree T'. Then, by part 1 of the IH, the max-player can achieve at least max-val(T'). By the definition of min-val,

min-val(T) ≤ max-val(T'). Thus the max-player can find a suitable move in any T' ∈ S.

3, 4) The arguments for 3 and 4 mirror those of 1, 2. Simply swap min and max everywhere.

b Now suppose the tree is not necessarily finite. The four statements now become

1) If the max-player is first to move in T, he can guarantee a payoff greater than max-val(T) − ε for any ε > 0.

2) If the max-player moves second in T, he can guarantee a payoff greater than min-val(T) − ε for any ε > 0

3) If the min-player moves first in T, he can guarantee a payoff less than min-val(T) + ε, for any ε > 0.

4) If the min-player moves second in T, he can guarantee a payoff less than max-val(T) + ε for any ε > 0

good!

Proof by structural induction, using these 4 statements as the IH

The four statements trivially hold for the one-node tree, using the same reasoning as for this step in part a.

Consider the case of a tree T constructed from a new root and a possibly-infinite set S of successors. Assume the IH holds for every element in S. Then it holds for T:

1) Max-val(T) is the lub of the min-vals of S. Let M = max-val(T) and ϵ some number greater than zero. There must be an element T' that has min-val(T') > $M - \epsilon$. (By contradiction, if there is no such element, then M cannot be the lub) Thus the max-player can move to T' and, by part 2 of the IH, be guaranteed payoff of min-val(T') which is > $M - \epsilon$

2) If the max-player moves second in T, then he is moving first in some subtree T' in the set S. By part 1 of the IH, he can achieve at least max-val(T') − ϵ. min-val(T) is the glb of the max-vals of T', so min-val(T) − ϵ must be less than or equal to max-val(T') − ϵ. Thus, the max-player can achieve at least min-val(T) − ϵ.

3,4) The arguments for 3 and 4 parallel those for 1, 2. Swap max-player and min-player, max-value and min-value, lub and glb.

This works in the unbounded case because max-value and min-value can be infinite, and everything still holds.

2. a.

n	f(n)
9996	9997
9997	9998
9998	9997
9999	9998
10000	9997

✓

b. f is defined by
f(x) = x − 3 for x ≥ 10000
f(x) = 9998 for n odd, n < 10000
✓ f(x) = 9997 for n even, n < 10000

Proof for parts b & c.
Let P(n) be the predicate that for a function f satisfying the given constraints, f(10000 − n) is given by the definition of f above.

By strong induction on n. P(n) is true for 0 ≤ n ≤ 4 using the cases found in part a. Then for any n ≥ 5, assume P(x) for all x < n.
P(n) is the statement that f(10000 − n) ✓ is given by the definition for f. This is true because f(10000 − n) = f(10000 − n + 5) = f(10000 − (n − 5)) which is true because P(n − 5) is assumed to be true. Thus P(n) holds for all n ∈ N and the above definition for f uniquely satisfies the given conditions.

3. a $\sum_{i=x}^{y} 2i+1 = \frac{(2y+1)+(2x+1)}{2}(y-x+1)$

$= (x+y+1)(y-x+1)$ ✓

b $\sum_{i=0}^{\infty}\sum_{j=1}^{n} \left(\frac{j}{j+2}\right)^i = \sum_{j=1}^{n}\sum_{i=0}^{\infty}\left(\frac{j}{j+2}\right)^i$

$= \sum_{j=1}^{n} \frac{1}{1-j/(j+2)} = \sum_{j=1}^{n} \frac{1}{\frac{j+2-j}{j+2}} = \sum_{j=1}^{n} \frac{j+2}{2}$

$= \frac{n}{2}\left(\frac{n+2}{2}+\frac{1+2}{2}\right) = n/2\left(\frac{n+2}{2}+\frac{3}{2}\right)$ ✓

c $\prod_{i=1}^{n} 2\cdot 4^{i} = \prod_{i=1}^{n} 2 \cdot \prod_{i=1}^{n} 4^{i} = 2^{n}\cdot 4^{\left(\sum_{i=1}^{n} i\right)}$

$= 2^{n}\cdot 4^{(n+1)n/2} = 2^{n}\cdot 4^{\left(\frac{n(n+1)}{2}\right)}$ oops

$= 4^{n/2}\, 4^{\frac{n(n+1)}{2}} = 4^{\frac{n(n+1)}{2}+\frac{n}{2}}$

$= 2^{n(n+1)+n} = 2^{n^2+2n}$ ✓

4 $\int_0^n (i+1)^{1/3}\,di$ = upper bound

$\int_0^n i^{1/3}\,di$ lower bound

$\int_0^n i^{1/3}\,di < S_n < \int_0^n (i+1)^{1/3}\,di$

$\int_0^n i^{1/3}\,di < S_n < \int_1^{n+1} u^{1/3}\,du$

$\frac{3}{4} i^{4/3}\Big|_0^n < S_n < \frac{3}{4} u^{4/3}\Big|_1^{n+1}$

✓ $3/4\, n^{4/3} < S_n < 3/4\,(n+1)^{4/3} - 3/4$

Thus $S_n \approx 3/4\, n^{4/3} \Rightarrow c = 3/4$

5. $\sum_{i=1}^{\infty} \frac{1}{i^2} = \sum_{i=1}^{n-1} \frac{1}{i^2} + \sum_{i=n}^{\infty} \frac{1}{i^2}$

Choose a value of n such that the second term has bounds that differ by 0.1:

$$\int_n^{\infty} \frac{1}{i^2}\,di \le \sum_{i=n}^{\infty} \frac{1}{i^2} \le \int_n^{\infty} \frac{1}{(i-1)^2}\,di$$

$$-\frac{1}{i}\Big|_n^{\infty} \le \sum_n^{\infty} \frac{1}{i^2} \le -\frac{1}{i}\Big|_{n-1}^{\infty}$$

$$0 + 1/n \le \sum_n 1/i^2 \le 0 + 1/_{n-1}$$

For the limits to differ by 0.1,

$1/_{n-1} - 1/n < 0.1 \Rightarrow n > 3.7$

Choose $n = 4$, since n must be an integer.

good! Then the sum is $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \sum_{i=4}^{\infty} \frac{1}{i^2}$

The sum of the first three terms is 1.361, and the sum from 4 to ∞ is bounded by $1/4$ and $1/3$. Thus the sum S satisfies:

$$1.361 + 1/4 \le S \le 1.361 + 1/3$$

$$\boxed{1.611 \le S \le 1.694}$$

6. $\binom{n}{n/2} = \frac{n!}{n/2!\,(n-n/2)!} = \frac{n!}{(n/2!)^2}$

Applying Stirling's approximation

$$= \frac{\sqrt{2\pi n}\,(n/e)^n}{2\pi n/2 \cdot (n/2e)^{n/2 \cdot 2}} = \frac{\sqrt{2}}{\sqrt{\pi n}} \frac{(n/e)^n}{(n/2e)^n}$$

$$= \sqrt{\frac{2}{\pi n}}\, 2^n = \sqrt{\frac{2}{\pi}}\, 2^n/\sqrt{n}$$

$\sqrt{\frac{2}{\pi}} \frac{2^n}{\sqrt{n}}$ is $\Theta(2^n/\sqrt{n})$:

$$\lim_{n \to \infty} \frac{2^n/\sqrt{n}}{\sqrt{2/\pi}\; 2^n/\sqrt{n}} = \sqrt{\frac{\pi}{2}}, \text{ a constant.}$$

6 cont. Thus $\binom{n}{n/2} = \Theta(2^n/\sqrt{n})$

7. a. $n + \ln n + (\ln n)^2$; n is the dominant term. $\boxed{\Theta(n)}$ ✓

b. $\lim_{n\to\infty} \frac{(n^2+2n-3)}{(n^2-1)} = 1$ $\boxed{\Theta(1)}$ ✓

c. $\sum_{i=0}^{n} 2^{2i+1} = 2 + 2^3 + 2^5 \dots 2^{2n+1}$

The sum of the first n terms is insignificant compared to the final term, so $j(n)$ grows as 2^{2n+1}

$\boxed{2^{f(n)} \text{ where } f(n) = \Theta(n)}$ ✓

d. $(2+\sin n)\,2^{(n+\sin n)}$ ranges between $(2+-1)2^{n-1}$ and $(2+1)2^{n+1}$.

Both are $\Theta(2^n)$ ✓, so $k(n) = \boxed{\Theta(2n)}$ $\Theta(2^n)$

e. Apply Stirling's approximation to $\ln((n^2)!)$:

$\ln\left(\sqrt{2\pi n^2}\left(\frac{n^2}{e}\right)^{n^2}\right) = \ln(n\sqrt{2\pi}) + n^2 \ln n^2 - n^2 \ln e$

$= \ln(n\sqrt{2\pi}) + n^2 \ln(n^2) - n^2$

$n^2 \ln(n^2)$ is the dominant equa the above

$2n^2 \ln(n) = \boxed{\Theta(n^2 \ln n)}$. ✓

(5/6)