Massachusetts Institute of Technology
6.042J/18.062J, Fall '02: Mathematics for Computer Science
Professor Albert Meyer and Dr. Radhika Nagpal

Solutions cover sheet
October 21

Student's Solutions to Problem Set 8

Your name: Dan Ports

Due date: October 28

Submission date: 10/28

Circle your TA: Adrian George Josh Karen Lee Min Nikos Tina

Collaboration statement: Circle one of the two choices and provide all pertinent info.

1. I worked alone and only with course materials.
2. I collaborated on this assignment with:

   got help from:[1]

   and referred to:[2]

DO NOT WRITE BELOW THIS LINE

Problem	Score
1	3
2	3
3	3
4	3
5	3
Total	15

[1]People other than course staff.

[2]Give citations to texts and material other than the class texts, handouts, and last term's bible.

Dan Ports
6.042 PS2.

1.a. The directed graph can be mapped to a $n \times n$ matrix whose entries represent edges. The dimension of this matrix is n^2, and each edge is either present or not, so the number of such graphs is $\boxed{2^{n^2}}$ ✓

b. Ignoring reflexive edges, there are $n(n+1)/2 - n = n^2/2 - n/2$ pairs of distinct nodes (ignoring the ordering of the pair). For each of these pairs there are three possibilities: no edges, the edge from the first to the second, and the edge from the second to the first; this results from the constraint. Thus there are $3^{(n^2/2 - n/2)}$ possible graphs. But we must also consider the n reflexive edges, each of which may or may not exist. This increases the number of possible graphs to $\boxed{3^{(n^2/2 - n/2)} \cdot 2^n}$ ✓

c. One of $2n$ output values must be chosen for each of n input values, so there are $\boxed{(2n)^n}$ possibilities. ✓

d. For the first input value there are $2n$ choices, for the second $2n-1$, down to $n+1$ for the last. This is a total of $\boxed{\frac{(2n)!}{n!}}$ functions ✓

d. Using the O() relation,

$B(n) < M(n) < U(n) < G(n)$

The $\Theta()$ relation never holds.

2. There are 10^6 total six-digit sequences. We subtract the number of sequences that contain 123 or 456. There are four possible starting positions for the 123 or 456, two options in the choice between 123 and 456, and 10 choices for each of the three remaining numbers. However, there are 4 numbers that are double-counted: 123123, 123456, 456123, and 456456. So the total is

$10^6 - (4 \cdot 2 \cdot 10^3 - 4) = \boxed{992,004}$

3. If some set of disks is to be placed onto one peg, there is only one allowable ordering of these disks on the peg, because the disks are of different sizes and no disk can rest on a smaller disk. Thus the problem can be reduced to finding the number of ways 32 disks can be placed on 3 pegs without regard for ordering. Each of the 32 disks must be assigned to one of the 3 pegs, so there are $\boxed{3^{32}}$ possible arrangements.

4. Let S_m be the mth partial sum $\sum_{i=1}^{m} a_i$. Then consider the set of partial sums $\{S_1, S_2, \dots S_n\}$. This set has n elements. It can be mapped to the set $\{0, 1, \dots n-1\}$ by taking the value of each partial sum mod n.

If there is a partial sum S_k that maps to the value zero, then we know that $n \mid S_k$, by definition of mod. Thus the value $S_k = \sum_{i=1}^{k} a_i$ is the sum we are looking for.

Otherwise there is no sum that maps to zero. Then all the sums map to values between 1 and $n-1$. Since there are n sums (pigeons) and a total of $n-1$ values they can map to (holes), we know by the pigeonhole principle that two sums must map to the same value. This means that they have the same value mod n. Call these sums S_k and S_ℓ and assume without loss of generality that $k < \ell$. Then by definition of mod,

$$n \mid (S_\ell - S_k) \Rightarrow n \mid \left(\sum_{i=1}^{\ell} a_i - \sum_{i=1}^{k} a_i\right)$$

$$\Rightarrow n \mid \sum_{i=k+1}^{\ell} a_i$$ which is what we were searching for.

5. To find square-free numbers, first find numbers that are multiples of squares. The set of squares less than 201 is $\{2^2, 3^2, 4^2, 5^2, \ldots 14^2\}$. But we can ignore all squares of non-primes because they are themselves products of squares of primes. This reduces the set to $\{2^2, 3^2, 5^2, 7^2, 11^2, 13^2\}$

First find the number of numbers strictly less than 201 that are divisible by one of these numbers:

$$\left\lfloor \frac{200}{2^2} \right\rfloor + \left\lfloor \frac{200}{3^2} \right\rfloor + \left\lfloor \frac{200}{5^2} \right\rfloor + \left\lfloor \frac{200}{7^2} \right\rfloor + \left\lfloor \frac{200}{11^2} \right\rfloor + \left\lfloor \frac{200}{13^2} \right\rfloor$$

$= 50 + 22 + 8 + 4 + 1 + 1$

$= 86$. By inclusion-exclusion, we must subtract the number of numbers divisible by more than one prime square. The numbers divisible by two prime squares are:

$$\left\lfloor \frac{200}{2^2 3^2} \right\rfloor + \left\lfloor \frac{200}{2^2 5^2} \right\rfloor + \left\lfloor \frac{200}{2^2 7^2} \right\rfloor = 5 + 3 + 1 = 8$$

(There are more terms, but they are all zero because every other product of 2 prime squares is greater than 200. Similarly, we must consider numbers that are divisible by three prime squares, but there are none of these because the smallest product $3^2 \cdot 5^2 \cdot 7^2$ is greater than 200).

By inclusion-exclusion there are $86 - 8 = 78$ non-square-free numbers. Therefore by the sum rule there are $200 - 78 = \boxed{122}$ square-free numbers.