

Problem Set 3

Problem 1:

a) Let X_i be independent geometrically distributed random variables with mean 2.

b) As in the Chernoff bound analysis, we note that the desired probability is equivalent to $\Pr [e^{tX} > e^{t(1+\epsilon)2n}]$.

By the Markov bound, this is less than $\frac{\mathbb{E}[e^{tX}]}{e^{t(1+\epsilon)2n}}$. Because X is the sum of independent X_i s, the expectation is $\prod_i \mathbb{E}[e^{tX_i}]$. Now note that

$$\begin{aligned} \mathbb{E}[e^{tX_i}] &= \sum_{j=1}^{\infty} e^{tj} \frac{1}{2} = \sum_{j=1}^{\infty} \left(\frac{e^t}{2}\right)^j \\ &= \frac{\frac{e^t}{2}}{1 - \frac{e^t}{2}} = \frac{2e^t}{2 - e^t} \end{aligned}$$

assuming $t < \ln 2$ so that the series converges. So

$$\Pr [X > (1 + \epsilon)2n] \leq \frac{\left(\frac{2e^t}{2 - e^t}\right)^n}{e^{t(1+\epsilon)2n}} = \frac{1}{((2 - e^t)e^{t+2t\epsilon})^n}$$

The TI-89 does not flinch in fear at this equation and tells us that the derivative is equal to zero at $t = \ln \frac{2\epsilon+1}{\epsilon+1}$. Thus this is the minimum. So

$$\Pr [X > (1 + \epsilon)2n] \leq \frac{1}{\left(\left(2 - \frac{2\epsilon+1}{\epsilon+1}\right) \left(\frac{2\epsilon+1}{\epsilon+1}\right)^{(1+2\epsilon)}\right)^n}$$

Problem 2:

Define a pivoting round to be “good” for a given item x if x is in a subproblem with size at most $9/10$ of the problem from which it was partitioned. So each “good” round reduces the size of the problem by $9/10$. Thus for any x , x will not be involved in more than $\log_{10/9} n$ good rounds, because this would reduce the size of a subproblem below 1.

For any subproblem of size m and any element x , the partition is a good pivoting round for x with probability at least $\frac{1}{2}$, since the pivot is chosen uniformly at random. Suppose x has rank r among the m elements. Assume without loss of generality that $r \leq \frac{m}{2}$ (if not, simply reverse the order of the ranking). Let p be the rank of the pivot. If $r < p \leq 9/10$ or $1/10 \leq p < r$, then this is a good pivoting round. This occurs with probability at least $1/2$. Moreover, the probability of any given pivoting round being good is independently above $\frac{1}{2}$ since they depend only on the relative rankings of the elements in the subproblem.

Let $f(n) = k \log_{10/9} n$. If $f(n)$ partitions are performed, then the expected number of good pivots is $\frac{k}{2} \log_{10/9} n$. Moreover, by the independence, we can apply the Chernoff bound and find that the

number of good pivots is less than $(1-\epsilon)\frac{k}{2}\log_{10/9} n$ with probability $e^{-\frac{\epsilon^2 \frac{k}{2} \log_{10/9} n}{2}} = e^{-\frac{\epsilon^2 k \log_{10/9} n}{4}}$. So the probability of having more than $\frac{k}{4}\log_{10/9} n$ good pivots ($\epsilon = 1/2$) is at least $e^{-\frac{-k \log_{10/9} n}{16}}$. If we choose $k = 16$, then the probability of having at least $4\log_{10/9} n$ is at least $e^{-\log_{10/9} n} = \left(\frac{1}{n}\right)^{\frac{1}{\log_{10/9} 9}} \approx \frac{1}{x^{9.49}}$.

Thus, with high probability, a sequence of $16\log_{10/9} n = \Theta(\log n)$ pivots gives at least $4\log_{10/9} n$ good pivots for any x , which we showed was enough to reduce the size of a subproblem below 1, and end the algorithm. So the algorithm has runtime $O(n \log n)$ with high probability.

Problem 3:

a) Consider the transpose permutation, mapping $a_i b_i$ to $b_i a_i$. We assume that the bit-fixing algorithm fixes bits from left to right. So if we fix some b , for all a_i , $a_i b$ will be converted to bb at some point in the process of being routed to ba_i . If the leftmost bit of a_i (call it a_1^0) differs from the corresponding bit of b (call it b^0 and the remainder $b^{1..n/2}$), then the $n/2 + 1$ th bit will be fixed and it will be routed through the link from bb to $ba^0 b^{1..n/2}$. There are $2^{n/2}$ choices for a_i in $a_i b$, and exactly half of these will have bit a_i^0 differing from bit b^0 . So $\frac{2^{n/2}}{2} = \Omega(\sqrt{N})$ packets are routed through this edge, and so $\Omega(\sqrt{N})$ routing steps are required.

b) Consider the transpose permutation again, mapping $a_i b_i$ to $b_i a_i$, and again fix some b . Now, however, the bits are fixed in random order, so it is not guaranteed that $a_i b$ will be converted to bb . Suppose a_i differs from b at k bits. Then it will be converted to bb if all of the k bits in the a_i half are fixed before the k bits in the b half. The probability of this is the probability that the first k bits chosen to be fixed from the $2k$ bits that differ are the k in the first half, i.e. $\frac{1}{\binom{2k}{k}}$. The number of bit strings that end in b and begin with some a_i that has k bits differing from b is $\binom{n/2}{k}$ since we choose the k bits that differ from the $n/2$ bits in a_i . Thus, combining these and applying the inequality, the expected number of packets that route through bb is

$$\sum_{k=1}^{n/2} \frac{\binom{n/2}{k}}{\binom{2k}{k}} \geq \sum_{k=1}^{n/2} \frac{\left(\frac{n/2}{k}\right)^k}{\left(\frac{2ek}{k}\right)^k} = \sum_{k=1}^{n/2} \left(\frac{n}{4ek}\right)^k$$

The value of the sum is certainly bounded below by any of its terms, so consider the term with $k = \lceil \frac{n}{8e} \rceil$. This reduces to $2^{\lceil \frac{n}{8e} \rceil}$ (to a constant factor), which is $2^{\Omega(n)}$. Since this is the expected number of packets that pass through bb , the routing algorithm must require this many steps in expectation.

Since the expectation is $2^{\Omega(n)}$, and this can be represented as the sum of Bernoulli random variables that indicate whether the i th packet has its bits flipped in the order that forces it to go through bb , we can apply the Chernoff bound. This tells us that the probability that the number of steps required deviates from its mean is small, and gives us a bound of $2^{\Omega(n)}$ with high probability.

Problem 4:

a) Let X_k that bin 1 contains k balls. It is given by the binomial distribution:

$$\Pr[X_k] = \binom{n}{k} \left(\frac{1}{n}\right)^k \left(1 - \frac{1}{n}\right)^{n-k}$$

which is bounded below by

$$\begin{aligned}\Pr[X_k] &\geq \binom{n}{k}^k \left(\frac{1}{n}\right)^k \left(\frac{n-1}{n}\right)^n \\ &\geq \frac{1}{k^k} \left(\frac{1}{2}\right)^n\end{aligned}$$

We want this probability to be at least $\frac{1}{\sqrt{n}}$. This gives us the constraint:

$$\begin{aligned}\Pr[X^k] &\geq \frac{1}{k^k} \left(\frac{1}{2}\right)^n \geq \frac{1}{\sqrt{n}} \\ k^k &\leq \sqrt{n} \\ 2k \log k \log n &\leq \frac{1}{2} \log n \\ k \log k &\leq \frac{3}{4} \log n\end{aligned}$$

Setting $k = \alpha \frac{\log n}{\log \log n}$,

$$\begin{aligned}\alpha \frac{\log n}{\log \log n} \log \left(\alpha \frac{\log n}{\log \log n} \right) &\leq \frac{3}{4} \log n \\ \alpha \frac{1}{\log \log n} (\log \alpha + \log \log n - \log \log \log n) &\leq \frac{3}{4} \\ \frac{\alpha \log \alpha}{\log \log n} + \alpha &\leq \frac{3}{4}\end{aligned}$$

For large n (greater than e^e), $\log \log n$ is greater than 1, so ignoring the denominator only strengthens the constraint. Thus we simply need to choose α small enough that $\alpha \log \alpha + \alpha \leq 3/4$. Note that k is still $\Omega\left(\frac{\log n}{\log \log n}\right)$. We have shown that the probability that bin 1 contains exactly k balls is at least $\frac{1}{\sqrt{2}}$, so this is certainly a bound on the probability of it having at least k balls.

Problem 5:

a) If there are αn balls at the beginning of a round, this is the problem of placing αn balls randomly in n bins. As the i th ball is added, there are at most $i - 1$ non-empty bins (if all balls previously went to empty bins; if there were earlier collisions than there are less). So the probability of the i th ball landing in a previously occupied bin is bounded above by $\frac{i-1}{n}$. By linearity of expectation, the expected number of balls landing in previously occupied bins is

$$\sum_{i=1}^{\alpha n} \frac{i-1}{n} = \frac{1}{n} \left(\sum_{i=1}^{\alpha n} i - \sum_{i=1}^{\alpha n} 1 \right) = \frac{1}{n} \left(\frac{(\alpha n)(\alpha n + 1)}{2} - \alpha n \right) = \frac{\alpha n(\alpha n - 1)}{2n} = \frac{\alpha^2 n - \alpha}{2} < \frac{\alpha^2 n}{2}$$

This probability is a sum of indicator random variables indicating whether the i th ball is placed in a previously occupied bin; thus, the Chernoff bound applies. By the Chernoff bound, the probability that the number of balls remaining is more than $\frac{2}{1.9}$ times the expectation, or $\frac{\alpha^2}{1.9}$, is $e^{-\frac{\alpha^2 n (\frac{2}{1.9})^2}{2}}$, so at most $\frac{\alpha^2}{1.9}$ balls remain with high probability.

b) We begin with n balls, i.e. $\alpha = 1$. At each stage, we have the recurrence

$$T(\alpha n) = 1 + T\left(\frac{\alpha^2 n}{1.9}\right)$$

Define α_1 to be 1 and $\alpha_n = \frac{\alpha_{n-1}^2}{2}$. Then the recurrence is

$$T(\alpha_i n) = 1 + T(\alpha_{i+1} n)$$

This recurrence continues until an i such that $\alpha_i < \frac{1}{n}$, at which point with high probability no balls remain. Taking the log of the recurrence for α_n , we find that $\log \alpha_n = \log \frac{\alpha_{n-1}^2}{2} = 2 \log \alpha_{n-1} - \log 2 \leq 2 \log \alpha_{n-1}$. The recurrence until $\log \alpha_i \leq -\log n$. Changing variables to $\beta_i = \log \alpha_i$ and $m = \log n$, we have $\beta_i \leq 2\beta_{i-1}$, a simple exponential recurrence that terminates once β_i reaches m . So the solution in terms of m is $O(\log m) = O(\log \log n)$.